INVARIANT AND STATIONARY MEASURES FOR THE $SL(2,\mathbb{R})$ ACTION ON MODULI SPACE

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Preliminary version

Abstract. We prove some ergodic-theoretic rigidity properties of the action of $SL(2,\mathbb{R})$ on moduli space. In this paper we show that any ergodic measure invariant under the action of the upper triangular subgroup of $SL(2,\mathbb{R})$ is supported on an invariant affine submanifold.

The main theorems are inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by Ratner's seminal work.

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1. Introduction

Suppose $g \geq 1$, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a partition of $2g-2$, and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1 \dots \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\alpha)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zo].

The space $\mathcal{H}(\alpha)$ admits an action of the group $SL(2,\mathbb{R})$ which generalizes the action of $SL(2,\mathbb{R})$ on the space $GL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of flat tori. In this paper we prove ergodic-theoretic rigidity properties of this action.

Let $\Sigma \subset M$ denote the set of zeroes of ω. Let $\{\gamma_1, \dots, \gamma_k\}$ denote a symplectic $\mathbb{Z}$-basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. We can define a map $\Phi : \mathcal{H}(\alpha) \to \mathbb{C}^k$ by

$$\Phi(M, \omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} w \right)$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \dots, \gamma_k\}$) is a local coordinate system on (M, ω). Alternatively, we may think of the cohomology class $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(\alpha)$. We will call these coordinates *period coordinates*.

We can consider the measure λ on $\mathcal{H}(\alpha)$ which is given by the pullback of the Lebesgue measure on $H^1(M, \Sigma, \mathbb{C}) \approx \mathbb{C}^k$. The measure λ is independent of the choice of basis $\{\gamma_1, \dots, \gamma_k\}$, and is easily seen to be $SL(2,\mathbb{R})$-invariant. We call λ the *Lebesgue* or the *Masur-Veech* measure on $\mathcal{H}(\alpha)$.

The area of a translation surface is given by

$$a(M, \omega) = \frac{i}{2} \int_M \omega \wedge \bar{\omega}.$$

A "unit hyperboloid" $\mathcal{H}_1(\alpha)$ is defined as a subset of translation surfaces in $\mathcal{H}(\alpha)$ of area one. The SL-invariant Lebesgue measure λ_1 on $\mathcal{H}_1(\alpha)$ is defined by disintegration of the Lebesgue measure λ on $\mathcal{H}_1(\alpha)$, namely

$$d\lambda = d\lambda_1 \, da$$

A fundamental result of Masur [Mas1] and Veech [Ve1] is that $\lambda_1(\mathcal{H}_1(\alpha)) < \infty$. In this paper, we normalize λ_1 so that $\lambda_1(\mathcal{H}_1(\alpha)) = 1$ (and so λ_1 is a probability measure).

For a subset $\mathcal{M}_1 \subset \mathcal{H}_1(\alpha)$ we write

$$\mathbb{R}\mathcal{M}_1 = \{(M, t\omega) \mid (M, \omega) \in \mathcal{M}_1, \quad t \in \mathbb{R}\} \subset \mathcal{H}(\alpha).$$

Definition 1.1. An ergodic $SL(2,\mathbb{R})$-invariant probability measure ν_1 on $\mathcal{H}_1(\alpha)$ is called *affine* if the following hold:

(i) The support $\mathcal{M}_1$ of ν_1 is an suborbitfold of $\mathcal{H}_1(\alpha)$. Locally $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ is given by a complex linear subspace defined over $\mathbb{R}$ in the period coordinates.

(ii) Let ν be the measure supported on $\mathcal{M}$ so that $d\nu = d\nu_1 da$. Then ν is an affine linear measure in the period coordinates on $\mathcal{M}$, i.e. it is (up to normalization) the restriction of the Lebesgue measure λ to the subspace $\mathcal{M}$.

Definition 1.2. We say that any suborbitfold $\mathcal{M}_1$ for which there exists a measure ν_1 such that the pair $(\mathcal{M}_1, \nu_1)$ satisfies (i) and (ii) an *affine invariant submanifold*.

We also consider the entire stratum $\mathcal{H}(\alpha)$ to be an (improper) affine invariant submanifold.

For many applications we need the following:

Proposition 1.3. *Any stratum $\mathcal{H}_1(\alpha)$ contains at most countably many affine invariant submanifolds.*

Proposition 1.3 is deduced as a consequence of some isolation theorems in [EMiMo]. This argument relies on adapting some ideas of G.A. Margulis to the Teichmüller space setting. Another proof is given by A. Wright in [Wr], where it is proven that affine invariant submanifolds are always defined over a number field.

The classification of the affine invariant submanifolds is complete in genus 2 by the work of McMullen [Mc1] [Mc2] [Mc3] [Mc4] [Mc5] and Calta [Ca]. In genus 3 or greater it is an important open problem. See [Mö1] [Mö2] [Mö3] [Mö4] [BoM] and [BaM] for some results in this direction.

1.1. **The main theorems.** Let

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R} \right\}, \quad \bar{N} = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

Let $r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and let $K = \{r_\theta \mid \theta \in [0, 2\pi)\}$. Then N, A and K are subgroups of $SL(2,\mathbb{R})$. Let $P = AN$ denote the upper triangular subgroup of $SL(2,\mathbb{R})$.

Theorem 1.4. *Let ν be any P-invariant probability measure on $\mathcal{H}_1(\alpha)$. Then ν is $SL(2,\mathbb{R})$-invariant and affine.*

The following (which uses Theorem 1.4) is joint work with A. Mohammadi and is proved in [EMiMo]:

Theorem 1.5. *Suppose $S \in \mathcal{H}_1(\alpha)$. Then, the orbit closure $\overline{PS} = \overline{SL(2,\mathbb{R})S}$ is an affine invariant submanifold of $\mathcal{H}_1(\alpha)$.*

For the case of strata in genus 2, the $SL(2,\mathbb{R})$ part of Theorem 1.4 and Theorem 1.5 were proved using a different method by Curt McMullen [Mc6].

The proof of Theorem 1.4 uses extensively entropy and conditional measure techniques developed in the context of homogeneous spaces (Margulis-Tomanov [MaT], Einsiedler-Katok-Lindenstrass [EKL]). Some of the ideas came from discussions with Amir Mohammadi. But the main strategy is to replace polynomial divergence by the "exponential drift" idea of Benoist-Quint [BQ].

Stationary measures. Let μ be a K-invariant compactly supported measure on $SL(2,\mathbb{R})$ which is absolutely continuous with respect to Lebesgue measure. A measure ν on $\mathcal{H}_1(\alpha)$ is called *μ-stationary* if $\mu * \nu = \nu$, where

$$\mu * \nu = \int_{SL(2,\mathbb{R})} (g_* \nu)\, d\mu(g).$$

Recall that by a theorem of Furstenberg [F1], [F2], restated as [NZ, Theorem 1.4], μ-stationary measures are in one-to-one correspondence with P-invariant measures. Therefore, Theorem 1.4 can be reformulated as the following:

Theorem 1.6. *Any μ-stationary measure on $\mathcal{H}_1(\alpha)$ is $SL(2,\mathbb{R})$ invariant and affine.*

Counting periodic trajectories in rational billiards. Let Q be a rational polygon, and let $N(Q,T)$ denote the number of cylinders of periodic trajectories of length at most T for the billiard flow on Q. By a theorem of H. Masur [Mas2] [Mas3], there exist c_1 and c_2 depending on Q such that for all $t > 1$,

$$c_1 e^{2t} \le N(Q, e^t) \le c_2 e^{2t}.$$

Theorem 1.4 and Proposition 1.3 together with some extra work (done in [EMiMo] imply the following "weak asymptotic formula" (cf. [AEZ]):

Theorem 1.7. *For any rational polygon Q, the exists a constant $c = c(Q)$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t N(Q, e^s) e^{-2s}\, ds = c.$$

The constant c in Theorem 1.7 is the Siegel-Veech constant (see [Ve2], [EMZ]) associated to the affine invariant submanifold $\mathcal{M} = \overline{SL(2,\mathbb{R})S}$ where S is the flat surface obtained by unfolding Q.

It is natural to conjecture that the extra averaging on Theorem 1.7 is not necessary, and one has $\lim_{t\to\infty} N(Q,e^t)e^{-2t} = c$. This can be shown if one obtains a classification of the measures invariant under the subgroup N of $SL(2,\mathbb{R})$. Such a result is in general beyond the reach of the current methods. However it is known in a few very special cases, see [EMS], [EMM], [CW] and [Ba].

Other applications to rational billiards. All the above theorems apply also to the moduli spaces of flat surfaces with marked points. Thus one should expect

applications to the "visibility" and "finite blocking" problems in rational polygons as in [HST]. It is likely that many other applications are possible.

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2. Outline of the paper

2.1. **Some notes on the proofs.** The theorems of §1.1 are inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by Ratner's seminal work. In particular, the analogues of Theorem 1.4 and Theorem 1.5 in homogeneous dynamics are due to Ratner [Ra4], [Ra5], [Ra6], [Ra7]. (For an introduction to these ideas, and also to the proof by Margulis and Tomanov [MaT] see the book [Mor].) The homogeneous analogue of the fact that P-invariant measures are $SL(2,\mathbb{R})$-invariant is due to Mozes [Moz] and is based on Ratner's work. All of these results are based in part on the "polynomial divergence" of the unipotent flow on homogeneous spaces.

However, in our setting, the dynamics of the unipotent flow (i.e. the action of N) on $\mathcal{H}_1(\alpha)$ is poorly understood, and plays no role in our proofs. The main strategy is to replace the "polynomial divergence" of unipotents by the "exponential drift" idea in the recent breakthrough paper by Benoist and Quint [BQ].

One major difficulty is that we have no apriori control over the Lyapunov spectrum of the geodesic flow (i.e. the action of A). By [AV] the Lyapunov spectrum is simple for the case of Lebesgue (i.e. Masur-Veech) measure, but for the case of an arbitrary P-invariant measure this is not always true, see e.g. [Fo2], [FoM].

In order to use the the Benoist-Quint exponential drift argument, we must show that the Zariski closure (or more precisely the algebraic hull, as defined by Zimmer [Zi2]) of the Kontsevich-Zorich cocycle is semisimple. The proof proceeds in the following steps:

Step 1. We use an entropy argument inspired by the "low entropy method" of [EKL] (using [MaT] together with some ideas from [BQ]) to show that any P-invariant measure ν on $\mathcal{H}_1(\alpha)$ is in fact $SL(2,\mathbb{R})$ invariant. We also prove Theorem 2.1 which gives control over the conditional measures of ν. This argument occupies §3-§13 and is outlined in more detail in §2.3.

Step 2. By some results of Forni (see Appendix A), for an $SL(2,\mathbb{R})$-invariant measure ν, the absolute cohomology part of the Kontsevich-Zorich cocycle $A : SL(2,\mathbb{R}) \times X \to Sp(2g,\mathbb{Z})$ is semisimple, i.e. has semisimple algebraic hull. For an exact statement see Theorem A.6.

Step 3. We pick an admissible measure μ on $SL(2,\mathbb{R})$ and work in the random walk setting as in [F1] [F2] and [BQ]. Let B denote the space of infinite sequences $g_0, g_1, \dots,$ where $g_i \in SL(2,\mathbb{R})$. We then have a skew product shift map $T : B \times X \to B \times X$ as in [BQ], so that $T(g_0, g_1, \dots; x) = (g_1, g_2, \dots; g_0^{-1}x)$. Then, we use (in Appendix C) a modification of the arguments by Guivarc'h and Raugi [GR1], [GR2], as presented by Goldsheid and Margulis in [GM, §4-5], and an argument of Zimmer (see [Zi1] or [Zi2]) to prove Theorem C.5 which states that the Lyapunov spectrum of T is always "semisimple", which means that for each $SL(2,\mathbb{R})$-irreducible component of the cocycle, there is an T-equivariant non-degenerate inner product on the Lyapunov subspaces of T (or more precisely on the successive quotients of the Lyapunov flag of T). This statement is trivially true if the Lyapunov spectrum of T is simple.

Step 4. We can now use the Benoist-Quint exponential drift method to show that the measure ν is affine. This is done in §14-§16. At one point, to avoid a problem with relative homology, we need to use a result, Theorem 14.3 about the isometric (Forni) subspace of the cocycle, which is proved in joint work with A. Avila and M. Möller [AEM].

Finally, we note that the proof relies heavily on various recurrence to compact sets results for the $SL(2,\mathbb{R})$ action, such as those of [EMa] and [Ath]. All of these results originate in the ideas of Margulis and Dani, [Mar1], [Dan1], [EMM1].

2.2. **Notational Conventions.** For $t \in \mathbb{R}$, let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let $A = \{g_t \ : \ t \in \mathbb{R}\}$, $N = \{u_t \ : \ t \in \mathbb{R}\}$. Let $P = AN$.

Let X denote the stratum $\mathcal{H}_1(\alpha)$ or a finite cover, see §4.3 below. Let $\tilde{X}$ denote the universal cover of X. Let $\pi : \tilde{X} \to X$ denote the natural map. A point of $\mathcal{H}(\alpha)$ is a pair (M, ω), where M is a compact Riemann surface, and ω is a holomorphic 1-form on M. Let Σ denote the set of zeroes of ω. The cohomology class of ω in the relative cohomology group $H^1(M, \Sigma, \mathbb{C}) \cong H^1(M, \Sigma, \mathbb{R}^2)$ is a local coordinate on $\mathcal{H}(\alpha)$ (see [Fo]). Let $V(x)$ denote a subspace of $H^1(M, \Sigma, \mathbb{R}^2)$. Then we denote

$$V[x] = \{y \in X \ : \ y - x \in V(x)\}.$$

This makes sense in a neighborhood of x.

Let $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ denote the natural map. Let

$$H^1_\perp(x) = \{v \in H^1(M, \Sigma, \mathbb{R}) \ : \ p(\operatorname{Re} x) \wedge p(v) = p(\operatorname{Im} x) \wedge p(v) = 0\}. \tag{2.1}$$

where we are considering the "real part map" Re and the "imaginary part map" Im as maps from $H^1(M, \Sigma, \mathbb{C}) \cong H^1(M, \Sigma, \mathbb{R}^2)$ to $H^1(M, \Sigma, \mathbb{R})$. Let

$$W(x) = \mathbb{R}(\operatorname{Im} x) \oplus H^1_\perp(x) \subset H^1(M, \Sigma, \mathbb{R})$$

and let $\pi_x^- : W(x) \to H^1(M, \Sigma, \mathbb{R})$ denote the map

$$\pi_x^-(c \operatorname{Im} x + v) = c \operatorname{Re} x + v \qquad c \in \mathbb{R}, v \in H^1_\perp(x). \tag{2.2}$$

We have $H^1(M, \Sigma, \mathbb{R}^2) = \mathbb{R}^2 \otimes H^1(M, \Sigma, \mathbb{R})$. For a subspace $V(x) \subset W(x)$, we write

$$V^+(x) = (1,0) \otimes V(x), \qquad V^-(x) = (0,1) \otimes \pi_x^-(V(x)). \tag{2.3}$$

Then $W^+[x]$ and $W^-[x]$ are the unstable and stable foliations for the action of g_t on X for $t > 0$.

We also use the notation,

$$\hat{W}^+(x) = (1,0) \otimes H^1(M, \Sigma, \mathbb{R}), \qquad \hat{W}^-(x) = (0,1) \otimes H^1(M, \Sigma, \mathbb{R}).$$

Starred Subsections. Some technical arguments are relegated to subsections marked with a star. These subsections can be skipped on first reading. The general rule is that no statement from a starred subsection is used in subsequent sections.

2.3. **Outline of the proof of Step 1.** The general strategy is based on the idea of additional invariance which was used in the proofs of Ratner [Ra4], [Ra5], [Ra6], [Ra7] and Margulis-Tomanov [MaT].

The aim of Step 1 is to prove the following:

Theorem 2.1. *Let ν be a P-invariant measure on the stratum X. Then ν is $SL(2,\mathbb{R})$-invariant. In addition, there exists an $SL(2,\mathbb{R})$-equivariant system of subspaces $\mathcal{L}(x) \subset H^1(M, \Sigma, \mathbb{R})$ such that for almost all x, the conditional measures of ν along $\hat{W}^+[x]$ are the Lebesgue measures along $\mathcal{L}^+[x]$, and the conditional measures of ν along $\hat{W}^-[x]$ are the Lebesgue measures along $\mathcal{L}^-[x]$.*

In the sequel, we will often refer to a (generalized) subspace $U^+(x) \subset W^+(x)$ on which we already proved that the conditional measure of ν is Lebesgue. The proof of Theorem 2.1 will be by induction, and in the beginning of the induction, $U^+[x] = Nx$. (Note: generalized subspaces are defined in §6).

Outline of the proof Theorem 2.1. Let ν be a P-invariant probability measure on X. Since ν is N-invariant, the conditional measure ν_{W^+} of ν along W^+ is non-trivial. This implies that the entropy of A is positive, and thus the conditional measure ν_{W^-} of ν along W^- is non-trivial. This implies that on a set of almost full measure, we can pick points q and q' in the support of ν such that q and q' are in the same leaf of W^- and $d(q, q') \approx 1$, see Figure 1.

Let $\ell > 0$ be a large parameter. Let $q_1 = g_\ell q$ and let $q_1' = g_\ell q'$. Then q_1 and q_1' are very close together. Pick $u \in U$ with $\|u\| \approx 1$, and consider the points uq_1 and uq_1'. These are no longer in the same leaf of W^-, and we expect them to diverge under the action of g_t as $t \to +\infty$. Let t be chosen so that $q_2 = g_t u q_1$ and $q_2' = g_t u q_1'$ be such that $d(q_2, q_2') \approx \epsilon$, where $\epsilon > 0$ is fixed.

Let

$$1 = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_n$$

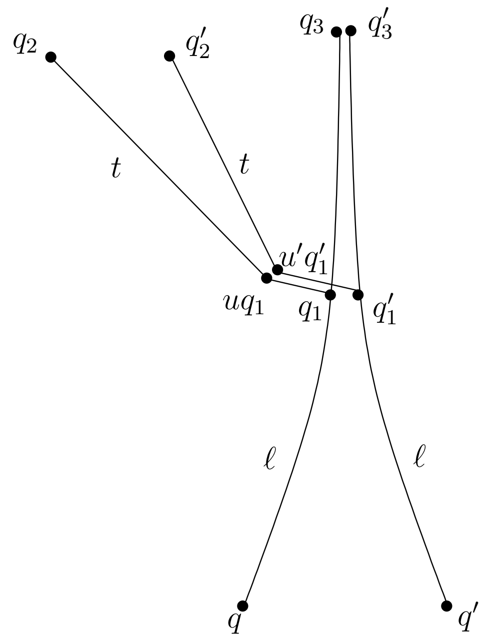

Figure 1. Outline of the proof of Theorem 2.1

denote the Lyapunov spectrum of the cocycle. Note that λ_0 corresponds to the unipotent direction inside the $SL(2,\mathbb{R})$ orbit.

We can choose q and q' so that $q - q'$ is orthogonal to the $SL(2,\mathbb{R})$ orbit of q. Then $q_2 - q_2'$ will also be orthogonal to the $SL(2,\mathbb{R})$ orbit of q_2. Generically one might expect that $q_2 - q_2'$ will approximately in the direction of the next highest Lyapunov exponent λ_1. Suppose for simplicity that this is indeed the case. For $x \in \mathcal{H}_1(\alpha)$, let $V_1^+(x) \subset W^+(x)$ be the Lyapunov subspace corresponding to the Lyapunov exponent λ_1 and let $f_1(x)$ denote the conditional measure of ν along $V_1^+[x]$.

Let $q_3 = g_s q_1$ and $q_3' = g_s q_1'$ where $s > 0$ is such that the amount of expansion along V_1^+ from q_1 to q_3 is equal to the amount of expansion along V_1^+ from uq_1 to q_2. Then, as in [BQ],

$$f_1(q_2) = (A)_* f_1(q_3), \text{ and } f_1(q_2') = (A')_* f_1(q_3'), \tag{2.4}$$

where A and A' are essentially the same bounded linear map. But q_3 and q_3' approach each other, so that

$$f_1(q_3) \approx f_1(q_3').$$

Hence

$$f_1(q_2) \approx f_1(q_2').$$

Taking a limit as $\ell \to \infty$ of the points q_2 and q_2' we obtain points $\tilde{q}_2$ and $\tilde{q}_2'$ in the same leaf of V_1^+ and distance ϵ apart such that

$$f_1(\tilde{q}_2) = f_1(\tilde{q}_2'). \tag{2.5}$$

This means that the conditional measure $f_1(\tilde{q}_2)$ is invariant under a shift of size approximately ϵ. Repeating this argument with $\epsilon \to 0$ we obtain a point p such that $f_1(p)$ is invariant under arbitrarily small shifts. This implies that the conditional measure $f_1(p)$ restricts to Lebesgue measure on some subspace U_{new} of V_1^+, which distinct from the orbit of N. Thus, we can enlarge U to be $U \oplus U_{new}$.

Technical Problem #1. The argument requires that all eight points q, q', q_1, q_1', q_2, q_2', q_3, q_3' belong to some "nice" set of almost full measure. Roughly, this can be arranged since the initial points q and q' can be chosen in a set of almost full measure (see §5). In §7 we show the maps used to choose the other points are essentially bilipshitz time changes of measure-preserving flows, so they take sets of almost full measure to sets of almost full measure. Thus, all eight points can be chosen in an a priori prescribed subset of almost full measure (see §12).

We discuss some of the strategy for dealing with this problem at the beginning of §5. The precise arguments are assembled in §12.

Technical Problem #2. Beyond the first step of the induction, the subspace $U^+(x)$ may not be locally constant as x varies along $W^+(x)$. This complication has a ripple effect on the proof. In particular, instead of dealing with the divergence of the points $g_t u q_1$ and $g_t u' q_1'$ we need to deal with the divergence of the affine subspaces $U^+[g_t u q_1]$ and $U^+[g_t u' q_1']$. As a first step, we project $U^+[g_t u' q_1']$ to the leaf of W^+ containing $U^+[g_t u q_1]$, to get a new affine subspace U'. One way to keep track of the relative location of $U^+ = U^+[g_t u' q_1']$ and U' is (besides keeping track of the linear parts of U^+ and U') to pick a transversal Z to U^+, and to keep track of the intersection of U' and Z, see Figure 2.

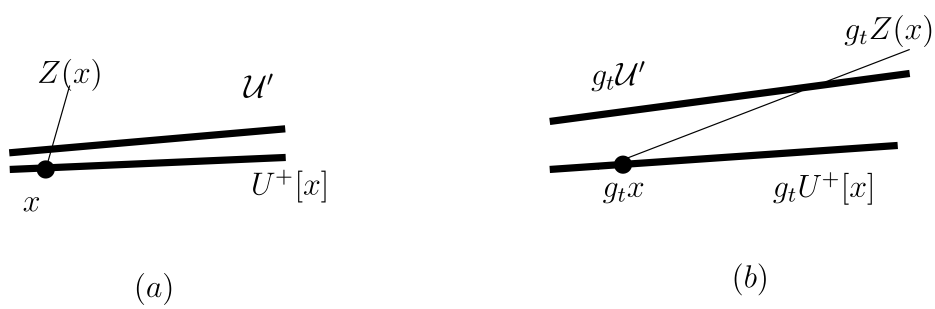

Figure 2.

(a) We keep track of the relative position of the subspaces $U^+[x]$ and $\mathcal{U}'$ in part by picking a transversal $Z(x)$ to $U^+[x]$, and noting the distance between $U^+[x]$ and $\mathcal{U}'$ along $Z[x]$.

(b) If we apply the flow g_t to the entire picture in (a), we see that the transversal $g_t Z[x]$ can get almost parallel to $g_t U^+[x]$. Then, the distance between $g_t U^+[x]$ and $g_t \mathcal{U}'$ along $g_t Z[x]$ may not be much larger then the distance between $g_t x \in g_t U^+[x]$ and the closest point in $g_t \mathcal{U}'$.

However, since we do not know at this point that the cocycle is semisimple, we cannot pick Z in a way which is invariant under the flow. Thus, we have no choice except to pick some transversal $Z(x)$ to $U^+(x)$ at ν-almost every point $x \in X$, and then deal with the need to change transversal.

It turns out that the formula for computing how $U' \cap Z$ changes when Z changes is non-linear (it involves inverting a certain matrix). However, we would really like to work with linear maps. This is done in two steps: first we show that we can choose the approximation U' and the transversals $Z(x)$ in such a way that changing transversals involves inverting a unipotent matrix. This makes the formula for changing transversals polynomial. In the second step, we embed the space of parameters of affine subspaces near $U^+[x]$ into a certain tensor power space $\mathbf{H}(x)$ so that the on the level of $\mathbf{H}(x)$ the change of transversal map becomes linear. The details of this construction are in §6.

Technical Problem #3. We do not have precise control over how q_2 and q_2' diverge. In particular the assumption that $q_2 - q_2'$ is nearly in the direction of $V_1^+(q_2)$ is not justified. Also we really need to work with $U^+[q_2]$ and $U^+[q_2']$. So let $\mathbf{v} \in \mathbf{H}(q_2)$ denote the vector corresponding to (the projection to $W^+(q_2)$ of) the affine subspace $U^+[q_2']$. (This vector $\mathbf{v}$ takes on the role of $q_2 - q_2'$). We have no a-priori control over the direction of $\mathbf{v}$ (even though we know that $\|\mathbf{v}\| \approx \epsilon$.). We need to define a *finite* collection of subspaces $\mathbf{E}_{[ij],bdd}(x)$ of $\mathbf{H}(x)$ such that

(a) By varying u (while keeping q_1, q_1', ℓ fixed) we can make sure that the vector $\mathbf{v}$ becomes close to one of the subspaces $\mathbf{E}_{[ij],bdd}$, and
(b) For a suitable choice of point $q_3 = q_{3,ij} = g_{s_{ij}} q_1$, the map
$$(g_t u g_{-s_{ij}})_* \mathbf{E}_{[ij],bdd}(q_3) \to \mathbf{E}_{[ij],bdd}(q_2)$$
is a bounded linear map.
(c) Also, for a suitable choice of point $q_3' = q_{3,ij}' = g_{s_{ij}'} q_1$, the map
$$(g_t u g_{-s_{ij}'})_* \mathbf{E}_{[ij],bdd}(q_3') \to \mathbf{E}_{[ij],bdd}(q_2')$$
is a bounded linear map.

For the precise conditions see Proposition 10.1 and Proposition 10.2. This construction is done in detail in §10.

The conditions (b) and (c) allow us to define in §11 conditional measures f_{ij} on $W^+(x)$ which are associated to each subspace $\mathbf{E}_{[ij],bdd}$. In fact the measures are supported on the points $y \in W^+[x]$ such that the affine subspace $U^+[y]$ maps to a vector in $\mathbf{E}_{[ij],bdd}(x) \subset \mathbf{H}(x)$.

Technical Problem #4. More careful analysis (see the discussion following the statement of Proposition 11.4) shows that the maps A and A' of (2.4) are not exactly the same. Then, when one passes to the limit $\ell \to \infty$ one gets, instead of (2.5),
$$f_{ij}(\tilde{q}_2) = P^+(\tilde{q}_2, \tilde{q}_2')_* f_{ij}(\tilde{q}_2')$$

where $P^+ : W^+(\tilde{q}_2) \to W^+(\tilde{q}_2')$ is a certain unipotent map (defined in §4.2). Thus the conditional measure $f_{ij}(\tilde{q}_2)$ is invariant under the composition of a translation of size ϵ and a unipotent map. Repeating the argument with $\epsilon \to 0$ we obtain a point p such that the conditional measure at p is invariant under arbitrarily small combinations of (translation + unipotent map). Thus does *not* imply that the conditional measure $f_{ij}(p)$ restricts to Lebesgue measure on some subspace of W^+, but it does imply that it is in the Lebesgue measure class along some polynomial curve in W^+. More precisely, for ν-a.e $x \in X$ there is a subgroup $U_{new} = U_{new}(x)$ of the affine group of $W^+(x)$ such that the conditional measure of $f_{ij}(x)$ on the polynomial curve $U_{new}[x] \subset W^+[x]$ is induced from the Haar measure on U_{new}. (We call such a set a "generalized subspace"). The exact definition is given in §6.

Thus, during the induction steps, we need to deal with generalized subspaces. This is not a very serious complication since the general machinery developed in §6 can deal with generalized subspaces as well as with ordinary affine subspaces. Also, at the end of the induction, we can show that we are dealing with a linear, i.e. non-generalized affine subspace (i.e. without loss of generality we may assume that U^+ consists of pure translations), see Proposition 6.9.

Completion of the proof of Theorem 2.1. Let $\mathcal{L}(x) \subset H^1(S, \Sigma, \mathbb{R})$ be the smallest subspace such that $\nu_{W^-(x)}$ is supported on $\mathcal{L}^-(x)$. The above argument can be iterated until we know the conditional measure $\nu_{W^+(x)}$ is Lebesgue on a subspace $\mathcal{U}^+[x]$, where $\mathcal{U}(x) \subset H^1(S, \Sigma, \mathbb{R})$ contains $\mathcal{L}(x)$. Then a Margulis-Tomanov style entropy comparison argument (see §13) shows that $\mathcal{U}(x) = \mathcal{L}(x)$, and the conditional measures along $\mathcal{L}^-(x)$ are Lebesgue. Since $\mathcal{U}^+(x)$ contains the orbit of the unipotent direction N, this implies that $\mathcal{L}^-(x)$ contains the orbit of the opposite unipotent direction $\bar{N} \subset SL(2, \mathbb{R})$. Thus, the conditional measure along the orbit of $\bar{N}$ is Lebesgue, which means that ν is $\bar{N}$-invariant. This, together with the assumption that ν is $P = AN$-invariant implies that ν is $SL(2, \mathbb{R})$-invariant, completing the proof of Theorem 2.1.

3. Hyperbolic properties of the geodesic flow

We recall the following standard fact:

Lemma 3.1 (Mautner Phenomenon). *Let ν be an ergodic P-invariant measure on a space Z. Then ν is A-ergodic.*

Proof. See e.g. [Moz]. □

Corollary 3.2. *For almost all $x \in X$, $W^+[x]$ and $W^-[x]$ are embedded, and do not contain any points from lower strata.*

Proof. Since W^- and W^+ are the stable and unstable foliations for the action of $g_t \in A$, this follows from the Poincare recurrence theorem. □

The bundles $H_{big}^{(+)}$, $H_{big}^{(-)}$ and $H_{big}^{(++)}$. and $H_{big}^{(--)}$. In this paper, we will need to deal with several bundles derived from the Hodge bundle. It is convenient to introduce a bundles $H_{big}^{(\pm)}$ so that every bundle we will need will be a subbundle of $H_{big}^{(+)}$ or $H_{big}^{(-)}$. Let $d \in \mathbb{N}$ be a large integer chosen later (it will be chosen in §6 and will depend only on the Lyapunov spectrum of the Kontsevich-Zorich cocycle). Let

$$H_{big}^{(+)}(x) = \bigoplus_{k=1}^{d} \bigoplus_{j=1}^{k} \left(\bigotimes_{i=1}^{j} W^+(x) \otimes \bigotimes_{l=1}^{k-j} (W^+(x))^* \right).$$

The flow g_t acts on the bundle H_{big} in the natural way. We denote this action by $(g_t)_*$. Let $H_{big}^{(++)}(x)$ denote the direct sum of the positive Lyapunov subspaces of $H_{big}^{(+)}(x)$.

Lemma 3.3. *The subspaces $H_{big}^{(++)}(x)$ are locally constant along $W^+[x]$, i.e. for almost all $y \in W^+[x]$ close to x we have $H_{big}^{(++)}(y) = H_{big}^{(++)}(x)$.*

Proof. Note that

$$H_{big}^{(++)}(x) = \left\{ v \in H_{big}^{(+)}(x) \quad : \quad \lim_{t \to \infty} \frac{1}{t} \log \frac{\|(g_{-t})_* v\|}{\|v\|} < 0 \right\}$$

Therefore, the subspace $H_{big}^{(++)}(x)$ depends only on the trajectory $g_{-t}x$ as $t \to \infty$. However, if $y \in W^+[x]$ then $g_{-t}y$ will for large t be close to $g_{-t}x$, and so in view of the locally linear structure, $(g_{-t})_*$ will be the same linear map on $H_{big}^{(+)}(x)$ and $H_{big}^{(+)}(y)$. This implies that $H_{big}^{(++)}(x) = H_{big}^{(++)}(y)$. $\square$

Similarly, let

$$H_{big}^{(-)}(x) = \bigoplus_{k=1}^{d} \bigoplus_{j=1}^{k} \left(\bigotimes_{i=1}^{j} W^-(x) \otimes \bigotimes_{l=1}^{k-j} (W^-(x))^* \right),$$

and let $H_{big}^{(--)}$ denote the direct sum of the negative Lyapunov subspaces of $H_{big}^{(-)}$. Then, the subspaces $H_{big}^{(--)}(x)$ are locally constant along $W^-[x]$.

The Avila-Gouëzel-Yoccoz norm. The Avila-Gouëzel-Yoccoz norm on the relative cohomology group $H^1(M, \Sigma, \mathbb{R})$ is described in §A. This then induces a norm on which we will denote by $\| \cdot \|_Y$ on $W^\pm$, and then, as the projective cross norm, also on $H_{big}^{(\pm)}$.

Choose a compact subset $K'_{thick} \subset X$ with $\nu(K'_{thick}) \geq 5/6$. Let $K_{thick} = \{x \in X \ : d(x, K'_{thick}) \leq 1\}$.

Lemma 3.4. *There exists $\alpha > 0$ such that the following holds: suppose $x \in X$ and $t > 0$ are such that the geodesic segment from x to $g_t x$ spends at least half the time in K_{thick}. Then,*

(a) *For all* $v \in W^-(x)$,
$$\|(g_t)_* v\|_Y \le e^{-\alpha t}\|v\|_Y.$$
(b) *For all all* $v \in W^+(x)$,
$$\|(g_t)_* v\|_Y \ge e^{\alpha t}\|v\|_Y.$$
(c) *For every* $\epsilon > 0$ *there exist a compact subset* $K''_{thick} \subset X$ *with* $\nu(K''_{thick}) > 1-\epsilon$ *and* $t_0 > 0$ *such that for* $x \in K''_{thick}$, $t > t_0$ *and all* $v \in H^{(++)}_{big}(x)$,
$$\|(g_t)_* v\|_Y \ge e^{\alpha t}\|v\|_Y.$$

Proof. Parts (a) and (b) follow from Theorem A.2. Part (c) follows immediately from the Osceledec multiplicative ergodic theorem. □

We also have the following simpler statement:

Lemma 3.5. *There exists* $N > 0$ *such that for all* $x \in X$, *all* $t \in \mathbb{R}$, *and all* $v \in H^{(+)}_{big}(x)$,
$$e^{-N|t|}\|v\|_Y \le \|(g_t)_* v\|_Y \le e^{N|t|}\|v\|_Y.$$
For $v \in W^+[x]$, *we can take* $N = 2$.

Proof. This follows immediately from Theorem A.2. □

Proposition 3.6. *Suppose* $C \subset X$ *is a set with* $\nu(C) > 0$, *and* $T_0 : C \to \mathbb{R}^+$ *is a measurable function which is finite a.e. Then we can find* $x_0 \in X$, *a subset* $C_1 \subset W^-[x_0] \cap C$ *and for each* $c \in C_1$ *a subset* $E^+[c] \subset W^+[c]$ *and a number* $t(c) > 0$ *such that if we let*
$$J_c = \bigcup_{0 \le t < t(c)} g_{-t} E^+[c],$$
then the following holds:

(a) $E^+[c]$ *is relatively open in* $W^+[c]$.
(b) $J_c \cap J_{c'} = \emptyset$ *if* $c \neq c'$.
(c) J_c *is embedded in* X, *i.e. if* $g_{-t}x = g_{-t'}x'$ *where* $x, x' \in E^+[c]$ *and* $0 \le t < t(c)$, $0 < t' < t(c)$ *then* $x = x'$ *and* $t = t'$.
(d) $\bigcup_{c \in C_1} J_c$ *is conull in* X.
(e) *For every* $c \in C_1$ *there exists* $c' \in C_1$ *such that* $g_{-t(c)}E^+[c] \subset E^+[c']$.
(f) $t(c) > T_0(c)$ *for all* $c \in C_1$.

Remark. All the construction in §3 will depend on the choice of C and T_0, but we will suppress this from the notation. The set C and the function T_0 will be finally chosen in Lemma 4.9.

The proof of Proposition 3.6 relies on the following:

Lemma 3.7. *Suppose $C \subset X$ is a set with $\nu(C) > 0$, and $T_0 : C \to \mathbb{R}^+$ is a measurable function which is finite a.e. Then we can find $x_0 \in X$, a subset $C_1 \subset W^-[x_0] \cap C$ and for each $c \in C_1$ a subset $E^+[c] \subset W^+[c]$ so that the following hold:*

(0) *$E^+[c]$ is a relatively open subset of $W^+[c]$.*
(1) *The set $E = \bigcup_{c \in C_1} E^+[c]$ is embedded in X, i.e. if $g_{-t}x = g_{-t'}x'$ where $x \in E^+[c]$, $x' \in E^+[c']$, $0 \le t < t(c)$, $0 < t' < t(c')$ then $c = c'$, $x = x'$ and $t = t'$.*
(2) *For some $\epsilon > 0$, $\nu(\bigcup_{t \in (0,\epsilon)} g_t E) > 0$.*
(3) *If $t > 0$ and $c \in C_1$ is such that $g_{-t}E^+[c] \cap E \neq \emptyset$, then $g_{-t}E^+[c] \subset E^+[c']$ for some $c' \in C_1$.*
(4) *Suppose t, c, c' are as in (3). Then $t > T_0(c)$.*

Proof. This proof is essentially identical to the proof of Lemma B.1, except that we need to take care that (4) is satified.

Choose $T_1 > 0$ so that if we let $C_4 = \{x \in C \ : \ T_0(x) < T_1\}$ then $\nu(C_4) > \nu(C)/2$. Let X_{per} denote the union of the periodic orbits of g_t. By ergodicity, $\nu(X_{per}) = 0$, and the same is true of the set $X'_{per} = \bigcup_{x \in X_{per}} W^+[x]$. Therefore there exists $x_0 \in C_4$ and subset $C_3 \subset W^-[x_0] \cap C_4$ with $\nu_{W^-[x_0]}(C_3) > 0$ such that for $x \in C_3$ and $0 < t < T_1$, $g_{-t}x \notin C_3$. Then, since C_3 is compact, we can find a small neighborhood $V^+ \subset W^+$ of the origin such that the set $\bigcup_{c \in C_3} V^+[c]$ is embedded in X and for $x \in \bigcup_{c \in C_3} V^+[c]$ and $0 < t < T_1$, $g_{-t}x \notin \bigcup_{c \in C_3} V^+[c]$.

Let $\alpha > 0$ be as in Lemma 3.4. Then, by Lemma 3.4 (c), without loss of generality we may assume that there exists $C_2 \subset C_3$ with $\nu_{W^+[x_0]}(C_2) > 0$ and $N > T_1$ such that for all $c \in C_2$ and all $T > N$,

$$|\{t \in [0,T] \ : \ g_{-t}c \in K'_{thick}\}| \ge T/2.$$

Then, for $c \in C_2$, $T > N$ and $x \in V^+[c]$,

$$|\{t \in [0,T] \ : \ g_{-t}x \in K_{thick}\}| \ge T/2.$$

Let

$$M = \sup \left\{ \frac{\|v\|_{Y,x}}{\|v\|_{Y,y}} \ : \ x \in V^+[c], y \in V^+[c], c \in C_2, v \in W^+(x) \right\}$$

Let $\alpha > 0$ be as in Lemma 3.4. and choose $N_1 > N$ such that $(M)^2 e^{-\alpha N_1} < 1/10$. Then, for $c \in C_2$, $x, y \in V^+[c]$ and $t > N_1$ such that $g_{-t}x \in \bigcup_{c \in C_2} V^+[c]$, in view of Lemma 3.4,

$$d(g_{-t}x, g_{-t}y) \le \frac{1}{10} d(x,y).$$

Now choose $C_1 \subset C_2$ with $\nu_{W^-(x_0)}(C_1) > 0$ so that if we let $Y = \bigcup_{c \in C_1} V^+[x]$ then $g_{-t}Y \cap Y = \emptyset$ for $0 < t < \max(T_1, N_1)$, in other words, the first return time to Y is at least $\max(T_1, N_1)$. (This can be done e.g. by Rokhlin's Lemma). Condition (4) now follows since $T_0(c) < T_1$ for all $c \in C_1$. The rest of the proof is essentially the same as the proof of Lemma B.1. $\square$

Proof of Proposition 3.6. For $x \in E$, let $t(x) \in \mathbb{R}^+$ be the smallest such that $g_{-t(x)}x \in E$. By property (3), the function $t(x)$ is constant on each set of the form $E^+[c]$. Let $F_t = \{x \in E \ : \ t(x) = t\}$. (We have $F_t = \emptyset$ if $t < N_1$). By property (2) and the ergodicity of g_{-t}, up to a null set,

$$X = \bigsqcup_{t>0} F_t.$$

Then properties (a)-(f) are easily verified. $\square$

Notation. Let $J[x]$ denote the set J_c containing x.

Lemma 3.8. *Suppose $y \in W^+[x] \cap J[x]$. Then for any $t > 0$,*

$$g_{-t}y \in J[g_{-t}x] \cap W^+[g_{-t}x].$$

Proof. This follows immediately from property (e) of Proposition 3.6. $\square$

Notation. Let

$$\mathfrak{B}_t[x] = g_{-t}(J[g_t x] \cap W^+[g_t x]).$$

Lemma 3.9.

(a) *For $t > t' \geq 0$, $\mathfrak{B}_t[x] \subseteq \mathfrak{B}_{t'}[x]$.*
(b) *Suppose $t \geq 0, t' \geq 0$ $x \in X$ $x' \in X$ are such that $\mathfrak{B}_t[x] \cap \mathfrak{B}_{t'}[x'] \neq \emptyset$. Then either $\mathfrak{B}_t[x] \supseteq \mathfrak{B}_{t'}[x']$ or $\mathfrak{B}_{t'}[x'] \supseteq \mathfrak{B}_t[x]$ (or both).*

Proof. Part (a) is a restatement of Lemma 3.8. For (b), without loss of generality, we may assume that $t' \geq t$. Suppose $y \in \mathfrak{B}_t[x] \cap \mathfrak{B}_{t'}[x']$. Then $y \in J[g_t x] \cap W^+[g_t x]$ and $y \in J[g_{t'}x'] \cap W^+[g_{t'}x']$. Since the $J[c]$ are pairwise disjoint, $J[g_t x] = J[g_{t'}x']$, and thus $\mathfrak{B}_t[x] = \mathfrak{B}_t[x']$. Now the statement follows from (a). $\square$

By construction, the sets $\mathfrak{B}_0[x]$ are the atoms of a measurable partition of X subordinate to W^+ (see Definition B.3). Then, let $\nu_{W^+(x)}$ denote the conditional measure of ν along the atom of the partition containing x.

Lemma 3.10. *Suppose $\nu(K) > 1 - \delta$. Then there exists a subset $K^* \subset K$ with $\nu(K^*) > 1 - \delta^{1/2}$ such that for any $x \in K^*$, and any $t > 0$,*

$$\nu_{W^+(x)}(K \cap \mathfrak{B}_t[x]) \geq (1 - \delta^{1/2})\nu_{W^+}(\mathfrak{B}_t[x]).$$

Proof. Let $E = K^c$, so $\nu(E) \leq \delta$. Let E^* denote the set of $x \in X$ such that there exists some $t \geq 0$ with

$$\nu_{W^+}(E \cap \mathfrak{B}_t[x]) \geq \delta^{1/2}\nu_{W^+}(\mathfrak{B}_t[x]). \tag{3.1}$$

It is enough to show that $\nu(E^*) \le \delta^{1/2}$. Let $t(x)$ be the smallest $t > 0$ so that (3.1) for x. Then the (distinct) sets $\{\mathfrak{B}_{t(x)}(x)\}_{x \in E^*}$ cover E^* and are pairwise disjoint by Lemma 3.9 (b). Let

$$F = \bigcup_{x \in E^*} \mathfrak{B}_{t(x)}(x).$$

Then $E^* \subset F$. For every leaf $W^+[y]$ of W^+ let $\Delta(y)$ denote the set of distinct sets $\mathfrak{B}_{t(x)}[x]$ where x varies over $W^+[y]$. Then, by (3.1)

$$\begin{aligned}\nu_{W^+}(F \cap W^+[y]) = \sum_{\Delta(y)} \nu_{W^+}(\mathfrak{B}_{t(x)}) \le \\ \le \delta^{-1/2} \sum_{\Delta(y)} \nu_{W^+}(E \cap \mathfrak{B}_{t(x)}[x]) \le \delta^{-1/2} \nu_{W^+}(E \cap W^+[y]).\end{aligned}$$

Integrating over y, we get $\nu(F) \le \delta^{-1/2}\nu(E)$. Hence,

$$\nu(E^*) \le \nu(F) \le \delta^{-1/2}\nu(E) \le \delta^{1/2}.$$

$\square$

The sets $\mathcal{B}$, $\mathcal{B}[x]$, $\mathcal{B}_t[x]$ and $\mathcal{B}(x)$. Let $\mathcal{B}_t[x] = U^+[x] \cap \mathfrak{B}_t[x]$. We will also use the notation $\mathcal{B}[x]$ for $\mathcal{B}_0[x]$.

For notational reasons, we will make the following construction: We fix a transversal $\hat{U}^+(x)$ to the stabilizer of $U^+(x) \cap Q_+(x)$ of x in $U^+(x)$. ($\hat{U}^+(x)$ need not be a group). Then, let

$$\mathcal{B}_t(x) = \{u \in \hat{U}^+(x) \;:\; ux \in \mathcal{B}_t[x]\}.$$

We pull pack the measure $|\cdot|$ from $\mathcal{B}_t[x]$ to $\mathcal{B}_t(x)$, and denote the measure again by $|\cdot|$. We also write $\mathcal{B}(x)$ for $\mathcal{B}_0(x)$. Finally, let $\mathcal{B} \subset U^+$ be the unit ball.

The same argument as Lemma 3.10 also proves the following:

Lemma 3.11. *Suppose $\nu(K) > 1 - \delta$ and $\theta' > 0$. Then there exists a subset $K^* \subset K$ with $\nu(K^*) > 1 - \delta/\theta'$ such that for any $x \in K^*$, and any $t > 0$,*

$$|K \cap \mathcal{B}_t[x]| \ge (1 - \theta')|\mathcal{B}_t[x]|,$$

and thus

$$|\{u \in \mathcal{B}_t(x) \;:\; ux \in K\}| \ge (1 - \theta')|\mathcal{B}_t(x)|.$$

4. General cocycle lemmas

4.1. **Lyapunov subspaces and flags.** Let $\mathcal{V}_i(x)$, $1 \le i \le k$ denote the Lyapunov subspaces of the Kontsevich-Zorich cocycle under the action of the geodesic flow g_t, and let λ_i, $1 \le i \le k$ denote the (distinct) Lyapunov exponents. Then we have for almost all $x \in X$,

$$H^1(M, \Sigma, \mathbb{R}) = \bigoplus_{i=1}^{k} \mathcal{V}_i(x)$$

and for all $v \in \mathcal{V}_i(x)$,

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \frac{\|(g_t)_* v\|}{\|v\|} = \lambda_i,$$

where $\|\cdot\|$ is any norm on $H^1(M, \Sigma, \mathbb{R})$ for example the Hodge norm defined in §A.1. By the notation $(g_t)_* v$ we mean the action of the geodesic flow (i.e. parallel transport using the Gauss-Manin connection) on the Hodge bundle $H^1(M, \Sigma, \mathbb{R})$. We note that the Lyapunov exponents of the geodesic flow (viewed as a diffeomorphism of X) are in fact $1 + \lambda_i$ and $-1 + \lambda_i$, $0 \le i \le k+1$. We have

$$1 = \lambda_1 > \lambda_1 > \cdots > \lambda_k = -1.$$

It is a standard fact that $\dim V_1 = \dim \hat{V}_1 = 1$, V_1 corresponds to the direction of the unipotent N and $\hat{V}_1$ corresponds to the direction of $\bar{N}$. Let $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ denote the natural map. If $x \in X$ denotes the pair (M, ω), then let

$$H^1_\perp(x) = \{\alpha \in H^1(M, \Sigma, \mathbb{R}) \ : \ p(\alpha) \wedge \mathrm{Re}\,(\omega) = p(\alpha) \wedge \mathrm{Im}\,(\omega) = 0\}.$$

Then

$$H^1_\perp(x) = \bigoplus_{i=2}^{k-1} \mathcal{V}_i(x).$$

We note that the subspaces $H^1_\perp(x)$ are equivariant under the $SL(2, \mathbb{R})$ action on X (since so is the subspace spanned by $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$). Since the cocycle preserves the symplectic form on $H^1_\perp$, we have

$$\lambda_{k+1-i} = -\lambda_i, \quad 1 \le i \le k.$$

Let

$$V_i(x) = \bigoplus_{j=1}^{i} \mathcal{V}_j(x), \qquad \hat{V}_i(x) = \bigoplus_{j=k+1-i}^{k} \mathcal{V}_j(x).$$

Then we have the Lyapunov flags

$$\{0\} = V_0(x) \subset V_1(x) \subset \cdots \subset V_k(x) = H^1(M, \Sigma, \mathbb{R})$$

and

$$\{0\} = \hat{V}_0(x) \subset \hat{V}_1(x) \subset \cdots \subset \hat{V}_k(x) = H^1(M, \Sigma, \mathbb{R}). \tag{4.1}$$

We record some simple properties of the Lyapunov flags:

Lemma 4.1.

(a) *The subspaces $V_i(x)$ are locally constant along $W^+[x]$, i.e. for almost all $y \in W^+[x]$ close to x we have $V_i(y) = V_i(x)$ for all $1 \le i \le k$.*
(b) *The subspaces $\hat{V}_i(x)$ are locally constant along $W^-[x]$, i.e. for almost all $y \in W^-[x]$ close to x we have $\hat{V}_i(y) = \hat{V}_i(y)$ for all $1 \le i \le k$.*

Proof. To prove (a), note that

$$V_i(x) = \left\{ v \in H^1_\perp(x) \ : \ \lim_{t\to\infty} \frac{1}{t} \log \frac{\|(g_{-t})_* v\|}{\|v\|} \le -\lambda_i \right\}$$

Therefore, the subspace $V_i(x)$ depends only on the trajectory $g_{-t}x$ as $t \to \infty$. However, if $y \in W^+[x]$ then $g_{-t}y$ will for large t be close to $g_{-t}x$, and so in view of the affine structure, $(g_{-s})_*$ will be the same linear map on $H^1(M,\Sigma,\mathbb{R})$ at $g_{-t}x$ and g_{-ty}, as in §3. This implies that $V_i(x) = V_i(y)$. The proof of property (b) is identical. $\square$

The action on W^+. We have $\hat{W}^+(x) = (1,0) \otimes H^1(M,\Sigma,\mathbb{R})$. For $v \in \hat{W}^+(x)$, we denote by $(g_t)_*$ the action of the geodesic flow (i.e. parallel transport using the Gauss-Manin connection followed by multiplication by e^t). All the results of §4.1 also apply to this action, except that the Lyapunov exponents λ_i are replaced by $1+\lambda_i$. We will usually consider the restiction of the map $(g_t)_*$ to $W^+(x) \subset \hat{W}^+(x)$ (see (2.3)).

4.2. **Equivariant measurable flat connections.** Let L be a subbundle of $H^{(++)}_{big}$. By an equivariant measurable flat W^+-connection on L we mean a measurable collection of linear "parallel transport" maps:

$$F(x,y) : L(x) \to L(y)$$

defined for ν-almost all $x \in X$ and $\nu_{W^+(x)}$ almost all $y \in W^+[x]$ such that

$$F(y,z)F(x,y) = F(x,z), \tag{4.2}$$

and

$$(g_t)_* \circ F(x,y) = F(g_t x, g_t y) \circ (g_t)_*. \tag{4.3}$$

For example, if $L = W^+(x)$, then the Gauss-Manin connection (which in period local coordinates is the identity map) is an equivariant measurable flat W^+ connection on W^+. However, there is another important equivariant measurable flat W^+-connection on W^+ which we describe below.

The maps $P^+(x,y)$ and $P^-(x,y)$. Recall that $\mathcal{V}_i^+(x) \subset W^+(x)$ are the Lyapunov subspaces for the flow g_t. Recall that the $\mathcal{V}_i^+(x)$ are not locally constant along leaves of W^+, but by Lemma 4.1, the subspaces $V_i^+(x) = \sum_{k=1}^i \mathcal{V}_i^+(x)$ are. Now suppose $y \in W^+[x]$. Any vector $v \in \mathcal{V}_i^+(x)$ can be written uniquely as

$$v = v' + v'' \qquad v \in \mathcal{V}_i^+(y), \quad v'' \in V_{i-1}^+(y).$$

Let $P_i^+(x,y) : \mathcal{V}_i^+(x) \to \mathcal{V}_i^+(y)$ be the linear map sending v to v'. Let $P^+(x,y)$ be the unique linear map which restricts to $P_i^+(x,y)$ on each of the subspaces $\mathcal{V}_i^+(x)$. We call $P^+(x,y)$ the "parallel transport" from x to y. The following is immediate from the definition:

Lemma 4.2. *Suppose $x, y \in W^+[z]$. Then*

(a) $P^+(x,y)\mathcal{V}_i^+(x) = \mathcal{V}_i^+(y)$.
(b) $P^+(g_t x, g_t y) = (g_t)_* \circ P^+(x,y) \circ (g_t^{-1})_*$.
(c) $P^+(x,y)V_i^+(x) = V_i^+(x)$. *If we identify* $W^+(x)$ *with* $W^+(y)$ *using the Gauss-Manin connection, then the map* $P^+(x,y)$ *is unipotent.*
(d) $P^+(x,z) = P^+(y,z) \circ P^+(x,y)$.

The statements (b) and (d) imply that the maps $P^+(x,y)$ define an equivariant measurable flat W^+-connection on W^+. This connection is in general different from the Gauss-Manin connection, and is only measurable.

If $y \in W^-[x]$, then we can define a similar map which we denote by $P^-(x,y)$. This yields an equivariant measurable flat W^--connection on W^-.

Clearly the connection $P^+(x,y)$ induces an equivariant measurable flat W^+-connection on any subbundle L of $H^{(++)}_{big}$. This connection preserves the Lyapunov subspaces of the g_t-action on L, as in Lemma 4.2 (a).

Equivariant measurable flat U^+-connections. Suppose $U^+[x] \subset W^+[x]$ is an g_t-equivariant system of algebraic subsets. By an equivariant measurable flat U^+-connection on a bundle $L \subset H^{(++)}_{big}$ we mean a measurable collection of linear maps $F(x,y) : L(x) \to L(y)$ satisfying (4.2) and (4.3), defined for ν-almost all $x \in X$ and $\nu_{U^+[x]}$-almost all $y \in U^+[x]$. In all the cases we will consider, the conditional measures along $\nu_{U^+[x]}$ are in the Lebesgue measure class (see §6).

4.3. **The Jordan Canonical Form of a cocycle.**

Passing to finite covers. We often get a situation where there is some finite set of subspaces $L_1(x), \dots, L_k(x)$ of $H^1_\perp$ which are permuted by the cocycle. Then there exists a finite extension of the action on which the subspaces L_i are invariant. See e.g. [ACO, §4] for details of this construction. Since the cocycle respects the affine structure, the extension may be identified with an action on a finite cover X' of X. (Here we are working in a measurable category and so all finite covers of a given degree are measurably isomorphic). Let ν' be an ergodic component of $\pi^{-1}(\nu)$ where $\pi : X' \to X$ is the covering map. By "passing to a finite cover" we mean replacing X by X' and ν by ν'.

Zimmer's Amenable reduction. The following is a general fact about linear cocycles over an action of $\mathbb{R}$ or $\mathbb{Z}$. It is often called "Zimmer's amenable reduction". We state it only for the cases which will be used.

Lemma 4.3. *Suppose* L_i *is a subbundle of* $H^{(++)}_{big}$. *(For example, we could have* $L_i(x) = \mathcal{V}_i(x)$*). Then, after possibly replacing* X *by a finite cover, there exists an invariant flag*

$$\{0\} = L_{i,0}(x) \subset L_{i,1}(x) \subset \cdots \subset L_{i,n_i}(x) = L_i(x), \tag{4.4}$$

and on each $L_{ij}(x)/L_{i,j-1}(x)$ there exists a nondegenerate quadratic form $\langle \cdot, \cdot \rangle_{ij,x}$ and a cocycle $\lambda_{ij} : X \times \mathbb{R} \to \mathbb{R}$ such that for all $u, v \in L_{ij}(x)/L_{i,j-1}(x)$,

$$\langle (g_t)_* u, (g_t)_* v \rangle_{ij, g_t x} = e^{\lambda_{ij}(x,t)} \langle u, v \rangle_{ij,x}. \tag{4.5}$$

Remark. The statement of Lemma 4.3 is the assertion that one can make a change of basis at each $x \in X$ so that in the new basis, the matrix of the cocycle restricted to L_i is of the form

$$\begin{pmatrix} C_{i,1} & * & \dots & * \\ 0 & C_{i,2} & \dots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \dots & C_{i,n_i} \end{pmatrix}, \tag{4.6}$$

where each $C_{i,j}$ is a conformal matrix (i.e. is the composition of an orthogonal matrix and a scaling factor λ_{ij}).

Proof of Lemma 4.3. See [ACO] (which uses many of the ideas of Zimmer). The statement differs slightly from that of [ACO, Theorem 5.6] in that we want the cocycle in each block to be conformal (and not just block-conformal). However, our statement is in fact equivalent because we are willing to replace the original space X by a finite cover. $\square$

4.4. **Covariantly constant subspaces.** The main result of this subsection is the following generalization of Lemma 4.1:

Proposition 4.4. *Suppose $L \subset H^{(++)}_{big}$ is an g_t-equivariant subbundle. We can write*

$$L(x) = \bigoplus_i L_i(x),$$

where $L_i(x)$ is the Lyapunov subspace corresponding to the Lyapunov exponent λ_i. Suppose there exists an equivariant flat measurable W^+-connection F on L, such that

$$F(x,y)L_i(x) = L_i(y), \tag{4.7}$$

and that $M \subset L$ is an g_t-equivariant subbundle. Then,

(a) *For almost all $y \in \mathfrak{B}_0[x]$,*

$$F(x,y)M(x) = M(y),$$

i.e. the subbundle M is locally covariantly constant with respect to the connection F.

(b) *For all i, the decomposition (4.4) of L_i is locally covariantly constant along W^+, i.e. for $\nu_{W^+(x)}$-almost all $y \in \mathfrak{B}_0[x]$, for all $i \in I$ and for all $1 \le j \le n_i$,*

$$L_{ij}(y) = F(x,y)L_{ij}(x). \tag{4.8}$$

Also, up to a scaling factor, the quadratic forms $\langle\cdot,\cdot\rangle_{i,j}$ are locally covariantly constant along W^+, i.e. for almost all $y \in \mathfrak{B}_0[x]$, and for $v, w \in L_{ij}(x)/L_{i,j-1}(x)$,

$$\langle F(x,y)v, F(x,y)w\rangle_{ij,y} = c(x,y)\langle v,w\rangle_{ij,x}. \tag{4.9}$$

Corollary 4.5. *Suppose $M \subset H^1(M,\Sigma,\mathbb{R})$ is an g_t-equivariant subbundle . Suppose also $V_{i-1}(x) \subseteq M(x) \subseteq V_i(x)$. Then, $M(x)$ is locally constant along $W^+(x)$.*

Proof of Corollary 4.5. By Lemma 4.1, $L(x) \equiv V_i(x)/V_{i-1}(x)$ is locally constant along $W^+[x]$. Let $F(x,y)$ denote the Gauss-Manin connection (i.e. the identity map) on $L(x)$. Note that the action of g_t on $L(x)$ has only one Lyapunov exponent, namely λ_i. Thus, (4.7) is trivially satisfied. Then, by Proposition 4.4 (a), $M(x)/V_{i-1}(x) \subset L(x)$ is locally constant along $W^+[x]$. Since $V_{i-1}(x)$ is also locally constant (by Lemma 4.1), this implies that $M(x)$ is locally constant. □

Remark. The proof of Proposition 4.4 is inspired by and is similar to the argument in Appendix C. In particular, it is also based on Lemma C.7.

To prove Proposition 4.4, we first prove the following:

Lemma 4.6. *Suppose L_i is either a subbundle or a quotient bundle of the Hodge bundle. Also suppose that there exists $\lambda_i \in \mathbb{R}$ such that for almost all $x \in X$ and all $v \in L_i(x)$,*

$$\lim_{t\to\infty} \frac{1}{t}\log\frac{\|(g_t)_* v\|}{\|v\|} = \lambda_i, \qquad \lim_{t\to-\infty} \frac{1}{t}\log\frac{\|(g_t)_* v\|}{\|v\|} = \lambda_i,$$

(so the Lyapunov spectrum of the cocycle on $L_i(x)$ consists of the single number λ_i). Then, for all j,

$$\int_X \lambda_{ij}(x,1)\, d\nu(x) = \lambda_i,$$

where the λ_{ij} are as in Lemma 4.3.

Proof. Let the L_{ij} be as in Lemma 4.3. Suppose $v \in L_{ij}(x) \setminus L_{i,j-1}(x)$ Then, using the basis as in (4.6), for $t > 0$,

$$\|(g_t)_* v\| \geq e^{\lambda_{ij}(x,t)}\|v\|.$$

Hence, for almost all $x \in X$,

$$\lambda_i = \lim_{t\to+\infty}\frac{1}{t}\log\frac{\|(g_t)_* v\|}{\|v\|} \geq \lim_{t\to\infty}\frac{1}{t}\lambda_{ij}(x,t) = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{\lfloor t\rfloor}\lambda_{ij}(g_k x, 1).$$

Integrating both sides over X we get

$$\lambda_i \geq \int_X \lambda_{ij}(x,1)\, d\nu(x).$$

To prove the opposite inequality, we consider the flow in the opposite direction. We have for $t > 0$,

$$\|(g_{-t})_* v\| \geq e^{\lambda_{ij}(x,-t)} \|v\|.$$

Hence, for almost all $x \in X$,

$$-\lambda_i = \lim_{t \to +\infty} \frac{1}{t} \log \frac{\|(g_{-t})_* v\|}{\|v\|} \geq \lim_{t \to \infty} \frac{1}{t} \lambda_{ij}(x,-t) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{\lfloor t \rfloor} \lambda_{ij}(g_{-k}x, -1).$$

Integrating both sides over X we get

$$-\lambda_i \geq \int_X \lambda_{ij}(x,-1)\, d\nu(x),$$

which implies

$$\lambda_i \leq \int_X \lambda_{ij}(x,1)\, d\nu(x).$$

□

Proof of Proposition 4.4. We now begin the proof of (a). We first make some preliminary reductions. Since $M(x)$ is g_t-equivariant,

$$M(x) = \bigoplus M_i(x), \qquad M_i(x) \subset L_i(x).$$

Thus, it is enough to show that

$$F(x,y) M_i(x) = M_i(y).$$

After applying Lemma 4.3 first to M_i and then to L_i/M_i, we get a flag

$$\{0\} \subset L_{i1}(x) \subset \cdots \subset L_{in}(x) = L_i(x), \tag{4.10}$$

such that $M_i = L_{ij}$ for some j, and such that (4.5) holds (and so the cocycle has the form (4.6)). Thus, it is enough to show that locally,

$$F(x,y) L_{ij}(x) = L_{ij}(y). \tag{4.11}$$

Suppose $x \in J_c$, where J_c is as in Proposition 3.6. Then the sets $\{g_{-t}c \;:\; 0 \leq t \leq t(c)\}$ and $\mathfrak{B}_0[x] = J_c \cap W^+[x]$ intersect at a unique point $x_0 \in X$. Then, we can replace the bundle $L(x)$ by $\tilde{L}(x) \equiv F(x,x_0)L(x)$. Then, for $y \in \mathfrak{B}_0[x]$,

$$\tilde{L}(y) = F(y,x_0)L(y) = F(y,x_0)F(x,y)L(x) = F(x,x_0)L(x) = \tilde{L}(x),$$

i.e. $\tilde{L}(x)$ is locally constant along $W^+(x)$. Also, by (4.3), the action of $(g_t)_*$ on $\tilde{L}$ is locally constant. Thus, without loss of generality, we may assume that F is locally constant (or else we replace L by $\tilde{L}$). Thus, in view of (4.11), it is enough to show that assuming the subspaces $L_i(x)$ are almost everywhere locally constant along W^+, the subspaces $L_{ij}(x)$ of (4.10) are also almost everywhere locally constant along W^+. Suppose that this does not hold. We will derive a contradiction.

The functions $L_{ij} : X \to Gr_{ij}$ where Gr_{ij} is the Grasmmannian of $\dim(L_{ij})$-planes are measurable. Therefore, for every $\epsilon > 0$ there is a subset K_ϵ with $\nu(K_\epsilon) > 1 - \epsilon$

such that the restriction of the L_{ij} to K_ϵ is uniformly continuous. By Lemma 3.10, there exists a subset $K_\epsilon^* \subset K_\epsilon$ with

$$\nu(K_\epsilon^*) > 1 - c(\epsilon),$$

so that for any $x \in K_\epsilon^*$ and any $t > 0$,

$$\nu_{W^+}(\mathfrak{B}_t[x] \cap K_\epsilon)| \geq (1 - c(\epsilon))\nu_{W^+}(\mathfrak{B}_t[x]),$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$.

For each $x \in X$ we fix a "standard basis" for L which varies continusly with x. Also, for each $x \in X$ there exists a basis β_x such that in this basis, the cocycle has the form (4.6). Let $H(x)$ be a linear map which takes the basis β_x to the standard basis. Then, we may assume that the function $x \to H(x)$ is continuous on K_ϵ and thus there exists a constant $C = C(\epsilon)$ such that

$$\max(\|H(x)\|, \|H(x)^{-1}\|) \leq C(\epsilon), \text{ for } x \in K_\epsilon. \tag{4.12}$$

We denote by $\hat{L}_{ij}(x)$ the complement of $L_{i,j-1}(x)$ in $L_{ij}(x)$ relative to the basis β_x.

Let $j \leq n$ be minimal such that for a positive measure set of $x \in X$ and a positive measure set of $y \in \mathfrak{B}_0[x]$, $L_{ij}(x) \neq L_{ij}(y)$. Let

$$L'(x) = L_{i,j-1}(x)$$

By the definition j, we have $L'(x) = L'(y)$ for almost all x and almost all $y \in \mathfrak{B}_0[x]$.

Since $L_{ij}(x) \neq L_{ij}(y)$, there exists $v \in \hat{L}_{ij}(y)$ such that $v \notin L_{ij}(x)$. We have $v \in V_i(x)$ (since $v \in L_{ij}(y) \subset V_i(y)$ and $V_i(y) = V_i(x)$ by Lemma 4.1. Then v has a component in $\hat{L}_{im}(x)$, where $L_{im}(x) \not\subset L_{ij}(x)$, i.e. $m > j$. We choose m to be minimal. Hence,

$$L'(x) = L_{i,j-1}(x) \subsetneq L_{ij}(x) \subseteq L_{i,m-1}(x) \subsetneq L_{im}(x).$$

It follows that there exist $\delta > 0$, $\sigma > 0$ and a set $E \subset X$ with $\nu(E) > 0$ such that for all $x \in E$ and for at least δ-fraction (with respect to ν_{W^+}) of $y \in \mathfrak{B}_0[x]$ there exists a unit vector $v(y) \in \hat{L}_{ij}(y)$ such that

$$v(y) = v_{im}(y) + w(y), \qquad v_{im}(y) \in \hat{L}_{im}(x), \qquad w(y) \in L_{i,m-1}(x), \tag{4.13}$$

and

$$\|v_{im}(y)\| \geq \sigma. \tag{4.14}$$

Let

$$\mathcal{D}^*(x) = \{t \in (-\infty, 0] \;\; : \;\; g_t x \in K_\epsilon^*\}.$$

Now choose $\epsilon > 0$ so that $2c(\epsilon) < \delta$.

Claim 4.7. *Suppose $x \in E \cap K_\epsilon^*$ and $t \in \mathcal{D}^*$. Then there exists $y = y(t) \in K_\epsilon$ such that $g_t y \in K_\epsilon$ and*

$$d(L_{ij}(g_t y), L_{ij}(g_t x)) \geq \sigma C(\epsilon)^{-1} e^{\lambda_{im}(x,t) - \lambda_{ij}(x,t)}. \tag{4.15}$$

Proof of claim. Since

$$g_t \mathfrak{B}_0[x] = \mathfrak{B}_{|t|}[g_t x],$$

and

$$\nu_{W^+}(\mathfrak{B}_{|t|}[g_t x] \cap K_\epsilon) \geq (1 - c(\epsilon))\nu_{W^+}(\mathfrak{B}_{|t|}[g_t x]),$$

we have that

$$\nu_{W^+}(\{y \in \mathfrak{B}_0[x] \ : \ g_t y \in K_\epsilon\}) \geq (1 - c(\epsilon))\nu_{W^+}(\mathfrak{B}_0[x]).$$

Also, since $x \in K_\epsilon^*$,

$$\nu_{W^+}(\mathfrak{B}_0[x] \cap K_\epsilon\}) \geq (1 - c(\epsilon))\nu_{W^+}(\mathfrak{B}_0[x]).$$

Since $2c(\epsilon) < \delta$, we can choose $y \in \mathfrak{B}_0[x]$ and $v \in \hat{L}_{ij}(y)$ such that (4.13) and (4.14) hold, and also

$$y \in K_\epsilon \text{ and } g_t y \in K_\epsilon. \tag{4.16}$$

Then, since $v \in \hat{L}_{ij}(y)$, and in view of (4.12) and (4.5),

$$C(\epsilon)^{-1} e^{\lambda_{ij}(y,t)} \leq d((g_t)_* v(y), L'(g_t y))) \leq C(\epsilon) e^{\lambda_{ij}(y,t)}.$$

Since the standard basis depends continuously on the base point, and in view of (4.12) and (4.16), the operator norm of the linear map taking β_x to β_y is bounded by a constant depending on ϵ. Similarly, since x and y stay together under g_t, $g_t x \in K_\epsilon$ and $g_t y \in K_\epsilon$, the operator norm of the linear map taking $\beta_{g_t x}$ to $\beta_{g_t y}$ is bounded by a constant depending on ϵ. Therefore,

$$C_1(\epsilon)^{-1} e^{\lambda_{ij}(x,t)} \leq e^{\lambda_{ij}(y,t)} \leq C_1(\epsilon) e^{\lambda_{ij}(x,t)} \tag{4.17}$$

Hence, since $L'(g_t y) = L'(g_t x)$,

$$C_2(\epsilon)^{-1} e^{\lambda_{ij}(x,t)} \leq d((g_t)_* v(y), L'(g_t x)) \leq C_2(\epsilon) e^{\lambda_{ij}(x,t)}. \tag{4.18}$$

But, in view of the form of the cocycle (4.6),

$$(g_t)_* v(y) = (g_t)_* v_{im}(y) + (g_t)_* w(y)$$

and

$$(g_t)_* v_{im}(y) = e^{\lambda_{im}(x,t)} O_{im}(x,t) v_{im}(y) + w' \in \hat{L}_{im}(g_t x) + w'$$

where $O_{im}(x,t)$ is an orthogonal matrix in the basis β_x, $w' \in L_{i,m-1}(g_t x)$, and

$$(g_t)_* w(y) \in L_{i,m-1}(g_t x)$$

Therefore, (because of the lower bound on the $\hat{L}_{im}$ component) and using (4.12),

$$d\Big((g_t)_* v(y), L_{ij}(g_t x)\Big) \geq d\Big((g_t)_* v(y), L_{i,m-1}(g_t x)\Big) \geq \sigma C(\epsilon)^{-1} e^{\lambda_{im}(x,t)}. \tag{4.19}$$

For $v \in V_i(x)$, let $\pi'_x(v)$ denote the point closest to the origin in the set $v + L'(x)$ (so in particular, since $L'(x) \subset L_{ij}(x)$, $\pi'_x(L_{ij}(x)) \subset L_{ij}(x)$). Thus, we can rewrite (4.19) as

$$d\biggl(\pi'_{g_t x}((g_t)_* v(y)), L_{ij}(g_t x)\biggr) \geq \sigma C(\epsilon)^{-1} e^{\lambda_{im}(x,t)}.$$

Also, in view of (4.18), we have

$$C_2(\epsilon)^{-1} e^{\lambda_{ij}(x,t)} \leq \|\pi'_{g_t x}((g_t)_* v(y))\| \leq C_2(\epsilon) e^{\lambda_{ij}(x,t)}.$$

Hence,

$$d\left(\frac{\pi'_{g_t x}((g_t)_* v(y))}{\|\pi'_{g_t x}((g_t)_* v(y))\|}, L_{ij}(g_t x)\right) \geq \sigma C(\epsilon)^{-1} e^{\lambda_{im}(x,t) - \lambda_{ij}(x,t)}. \tag{4.20}$$

Since $L'(g_t x) = L'(g_t y)$ we have $\pi'_{g_t y}((g_t)_* v(y)) = \pi'_{g_t x}((g_t)_* v(y))$. Hence

$$d\left(\frac{\pi'_{g_t y}((g_t)_* v(y))}{\|\pi'_{g_t y}((g_t)_* v(y))\|}, L_{ij}(g_t x)\right) \geq \sigma C(\epsilon)^{-1} e^{\lambda_{im}(x,t) - \lambda_{ij}(x,t)}.$$

Since $\pi'_{g_t y}((g_t)_* v(y)) \in L_{ij}(g_t y)$, this completes the proof of the claim. □

We now continue the proof of Proposition 4.4. For $t \in \mathcal{D}^*(x)$, let $y = y(t)$ be as in the claim. Then, as $t \to -\infty$ in $\mathcal{D}^*(x)$, $d(g_t x, g_t y) \to 0$. Then, since $g_t x \in K_\epsilon$ and $g_t y \in K_\epsilon$,

$$d(L_{ij}(g_t y), L_{ij}(g_t x)) \to 0. \tag{4.21}$$

Then, (4.21) and (4.15) imply that for all $x \in E$,

$$\lambda_{ij}(x,t) - \lambda_{im}(x,t) \to \infty, \qquad \text{as } t \to -\infty \text{ in } \mathcal{D}^*(x).$$

Let $T = g_{-1}$ denote the time -1 map of the geodesic flow. Since λ_{ij} and λ_{im} are cocycles, for $t \in \mathbb{Z}$,

$$\lambda_{ij}(x,t) = \sum_{n=0}^{t-1} \lambda_{ij}(T^n x, 1),$$

Therefore, for all $x \in E \cap K_\epsilon^*$,

$$\liminf_{n \to \infty} \left\{ \sum_{r=0}^{n-1} \lambda_{ij}(T^r x, 1) - \lambda_{im}(T^r x, 1) \;\;:\;\; T^n x \in K_\epsilon^* \right\} = \infty. \tag{4.22}$$

Since for almost all $x \in X$, $T^n x \in E \cap K_\epsilon^*$ for some $n \in \mathbb{N}$, (4.22) holds for almost all $x \in X$. Therefore, by Lemma C.7,

$$\int_X \lambda_{ij}(x,1)\, d\nu(x) > \int_X \lambda_{im}(x,1)\, d\nu(x).$$

This contradicts Lemma 4.6, and completes the proof of (a), and also of (4.8).

We now prove (4.9). Let $K \subset K_\epsilon$ denote a compact subset with $\nu(K) > 0.9$ where $\langle \cdot, \cdot \rangle_{ij}$ is uniformly continuous. Consider the points $g_t x$ and $g_t y$, as $t \to -\infty$. Then $d(g_t x, g_t y) \to 0$. Let

$$v_t = e^{-\lambda_{ij}(x,t)}(g_t)_* v, \quad w_t = e^{-\lambda_{ij}(x,t)}(g_t)_* w.$$

Then, by Lemma 4.3, we have

$$\langle v_t, w_t \rangle_{ij, g_t x} = \langle v, w \rangle_{ij,x}, \quad \langle v_t, w_t \rangle_{ij, g_t y} = c(x,y,t) \langle v, w \rangle_{ij,y}. \tag{4.23}$$

where $c(x,y,t) = e^{\lambda_{ij}(x,t) - \lambda_{ij}(y,t)}$.

Now take a sequence $t_k \to \infty$ with $g_{t_k} x \in K$, $g_{t_k} y \in K$ (such a sequence exists for ν-a.e. x and y with $y \in \mathfrak{B}_0[x]$). Then in view of (4.17), $c(x,y,t_k)$ is bounded between two constants. Also,

$$\langle v_{t_k}, w_{t_k} \rangle_{ij, g_{t_k} x} \to \langle v_{t_k}, w_{t_k} \rangle_{ij, g_{t_k} y}.$$

Now the equation (4.9) follows from (4.23). $\square$

The proof also shows the following:

Remark 4.8. Proposition 4.4 applied also to U^+-connections, provided the measure along $U^+[x]$ is in the Lebesgue measure class, and provided that in the statement, the set $\mathfrak{B}_0[x]$ is replaced by $\mathcal{B}[x] = \mathfrak{B}_0[x] \cap U^+[x]$.

4.5. **Dynamically defined norms.** In this subsection we define a norm on $\| \cdot \|$ on $H^{(++)}_{big}$, which has some advantages over the AGY norm $\| \cdot \|_Y$.

The function $\Xi(x)$. Let

$$\Xi^+(x) = \sup_{ij} \sup \left\{ \langle v, v \rangle_{ij,x}^{1/2}, \ : \ v \in L_{ij}(x)/L_{i,j-1}(x), \ \|v\|_{Y,x} = 1 \right\},$$

and let

$$\Xi^-(x) = \inf_{ij} \inf \left\{ \langle v, v \rangle_{ij,x}^{1/2}, \ : \ v \in L_{ij}(x)/L_{i,j-1}(x), \ \|v\|_{Y,x} = 1 \right\}.$$

Let

$$\Xi(x) = \Xi^+(x)/\Xi^-(x). \tag{4.24}$$

We have $\Xi(x) \geq 1$ for all $x \in X$.

Lemma 4.9. *Fix $\epsilon > 0$ smaller than $\min_i |\lambda_i|$, and smaller than $\min_{ij} |\lambda_i - \lambda_j|$, where the λ_i are the Lyapunov exponents of $H^{(++)}_{big}$. There exists a compact subset $C \subset X$ with $\nu(C) > 0$ and function $T_0 : C \to \mathbb{R}^+$ with $T_0(x) < \infty$ for ν a.e. $x \in C$ such that the following hold:*

(a) *There exists $\sigma > 0$ such that for all $c \in C$, and any subset S of the Lyapunov exponents,*

$$d\Big(\bigoplus_{i \in S} L_i(c), \bigoplus_{j \notin S} L_j(c)\Big) \geq \sigma,$$

where $d(\cdot, \cdot)$ denotes (any) distance between subspaces.

(b) *There exists* $M' > 1$ *such that for all* $c \in C$, $\Xi(c) \le M'$.

(c) *For all* $c \in C$ *and for all* $t > T_0(c)$, *for any subset* S *of the Lyapunov spectrum,*

$$d(\bigoplus_{i\in S} L_i(g_{-t}c), \bigoplus_{j\notin S} L_j(g_{-t}c)) \ge e^{-\epsilon t},$$

where $L_i(c) = \mathcal{V}_i(H^{(++)}_{big}(c))$. *Hence, for all* $c \in C$ *and all* $t > T_0(c)$ *and all* $c' \in C \cap W^+[g_{-t}c]$ *with* $d(c, c') < 1$,

$$\rho_1^{-1} e^{-\epsilon t} \le \|P^+(g_{-t}c, c')\|_Y \equiv \sup_{v\neq 0} \frac{\|P^+(g_{-t}c, c')v\|_{Y,c'}}{\|v\|_{Y,g_{-t}c}} \le \rho_1 e^{\epsilon t}, \tag{4.25}$$

where $\rho_1 = \rho_1(M', \sigma, \sup\{\|v\|_{Y,x}/\|v\|_{Y,y} \;:\; x, y \in C, d(x, y) < 1\})$.

(d) *There exists* $\rho > 0$ *such that for all* $t > T_0(x)$, *for all* $c \in C$, *for all* i *and all* $v \in L_i(c)$,

$$e^{-(\lambda_i+\epsilon)t}\rho_1\rho^2\|v\|_{Y,c} \le \|g_{-t}v\|_{Y,g_{-t}c} \le \rho_1^{-1}\rho^{-2}e^{-(\lambda_i-\epsilon)t}\|v\|_{Y,c}. \tag{4.26}$$

Proof. Parts (a) and (b) hold since the inverse of the angle between Lyapunov subspaces and since the ratio of the norms are finite a.e., therefore bounded on a set of almost full measure. To see (c), note that by the Osceledec multiplicative ergodic theorem, [KH, Theorem S.2.9 (2)] for ν-a.e. $x \in X$,

$$\lim_{t\to\infty} \frac{1}{t} \log \sin |\angle(\bigoplus_{i\in S} L_i(g_{-t}x), \bigoplus_{j\notin S} L_j(g_{-t}x)) = 0.$$

Also, (d) follows immediately from the multiplicative ergodic theorem. $\square$

We now choose the set C and the function T_0 of Proposition 3.6 and Lemma 3.7 to be as in Lemma 4.9.

The inner products $\langle\cdot,\cdot\rangle_{ij}$ on $E^+[c]$. Note that the inner products $\langle\cdot,\cdot\rangle_{ij}$ and the $\mathbb{R}$-valued cocycles λ_{ij} of Lemma 4.3 are not unique, since we can always multiply $\langle\cdot,\cdot\rangle_{ij,x}$ by a scalar factor $c(x)$, and then replace $\lambda_{ij}(x,t)$ by $\lambda_{ij}(x,t) + \log c(g_t x) - \log c(x)$. In view of (4.9) in Proposition 4.4 (b), we may (and will) use this freedom to make $\langle\cdot,\cdot\rangle_{ij,x}$ constant on each set $E^+[c]$, where $E^+[c]$ is as in §3.

The inner product $\langle\cdot,\cdot\rangle_x$ on $E^+[c]$. Let

$$\{0\} = V_0 \subset V_1 \subset \dots \tag{4.27}$$

be the Lyapunov flag for $H^{(++)}_{big}$, and for each i, let

$$V_{i-1} = V_{i0} \subset V_{i1} \subset \dots V_{i,n_i} = V_i \tag{4.28}$$

be a maximal invariant refinement.

Let $L_i = \mathcal{V}_i(H^{(++)}_{big})$ denote the Lyapunov subspaces for $H^{(++)}_{big}$. Then we have a maximal invariant flag

$$\{0\} = L_{i,0} \subset L_{i,1} \subset \cdots \subset L_{i,n_i} = L_i,$$

where $L_{ij} = L_i \cap V_{ij}$.

Let c, $E^+[c]$ be as in §3. By Lemma 4.9 (b), we can (and do) rescale the inner products $\langle \cdot, \cdot \rangle_{ij,c}$ so that after the rescaling, for all $v \in L_{ij}(c)/L_{i,j-1}(c)$,

$$(M')^{-1} \|v\|_{Y,c} \le \langle v, v \rangle_{ij,c}^{1/2} \le M' \|v\|_{Y,c},$$

where $\| \cdot \|_{Y,c}$ is the AGY norm at c and $M' > 1$ is as in Lemma 4.9. We then choose $L'_{ij}(c) \subset L_{ij}(c)$ to be a complementary subspace to $L_{i,j-1}(c)$ in $L_{ij}(c)$, so that for all $v \in L_{i,j-1}(c)$ and all $v' \in L'_{ij}(c)$,

$$\|v + v'\|_{Y,c} \ge \rho'' \max(\|v\|_{Y,c}, \|v'\|_{Y,c}),$$

and $\rho'' > 0$ depends only on the dimension.

Then,

$$L'_{ij}(c) \cong L_{ij}(c)/L_{i,j-1}(c) \cong V_{ij}(c)/V_{i,j-1}(c).$$

Let $\pi_{ij} : V_{ij} \to V_{ij}/V_{i,j-1}$ be the natural quotient map. Then the restriction of π_{ij} to $L'_{ij}(c)$ is an isomorphism onto $V_{ij}(c)/V_{i,j-1}(c)$.

We can now define for $u, v \in H^{(++)}_{big}(c)$,

$$\langle u, v \rangle_c \equiv \sum_{ij} \langle \pi_{ij}(u_{ij}), \pi_{ij}(v_{ij}) \rangle_{ij,c},$$
$$\text{where } u = \sum_{ij} u_{ij},\ v = \sum_{ij} v_{ij},\ u_{ij} \in L'_{ij}(c),\ v_{ij} \in L'_{ij}(c).$$

In other words, the distinct $L'_{ij}(c)$ are orthogonal, and the inner product on each $L'_{ij}(c)$ coincides with $\langle \cdot, \cdot \rangle_{ij,c}$ under the identification π_{ij} of $L'_{ij}(c)$ with $V_{ij}(c)/V_{i,j-1}(c)$.

We now define, for $x \in E^+[c]$, and $u, v \in H^{(++)}_{big}(x)$

$$\langle u, v \rangle_x \equiv \langle P^+(x,c)u, P^+(x,c)v \rangle_c,$$

where $P^+(\cdot, \cdot)$ is the connection defined in §4.2. Then for $x \in E^+[c]$, the inner product $\langle \cdot, \cdot \rangle_x$ induces the inner product $\langle \cdot, \cdot \rangle_{ij,x}$ on $V_{ij}(x)/V_{i,j-1}(x)$.

Symmetric space interpretation. We want to define the inner product $\langle \cdot, \cdot \rangle_x$ for any $x \in J[c]$ by interpolating between $\langle \cdot, \cdot \rangle_c$ and $\langle \cdot, \cdot \rangle_{c'}$, where c' is such that $g_{-t(c)}c \in E^+[c']$. To define this interpolation, we recall that the set of inner products on a vector space V is canonically isomorphic to $GL(V)/SO(V)$, where $GL(V)$ is the general linear group of V and $SO(V)$ is the subgroup preserving the inner product on V. In our case, $V = H^{(++)}_{big}(c)$ with the inner product $\langle \cdot, \cdot \rangle_c$.

Let K_c denote the subgroup of $GL(H^{(++)}_{big}(c))$ which preserves the inner product $\langle \cdot, \cdot \rangle_c$. Let $\mathcal{Q}$ denote the parabolic subgroup of $GL(H^{(++)}_{big}(c))$ which preserves the flags (4.27) and (4.28), and on each succesive quotient $V_{ij}(c)/V_{i,j-1}(c)$ preserves $\langle \cdot, \cdot \rangle_{ij,c}$. Let

$A'K_c$ denote the point in $GL(H^{(++)}_{big}(c))/K_c$ which represents the inner product $\langle \cdot, \cdot \rangle_{c'}$, i.e.

$$\langle u, v \rangle_{c'} = \langle A'u, A'v \rangle_c.$$

Then, since $\langle \cdot, \cdot \rangle_{c'}$ induces the inner products $\langle \cdot, \cdot \rangle_{ij,c'}$ on the space $V_{ij}(c')/V_{i,j-1}(c')$ (which is the same as $V_{ij}(c)/V_{i,j-1}(c)$), we may assume that $A' \in \mathcal{Q}$.

Let $N_{\mathcal{Q}}$ be the normal subgroup of $\mathcal{Q}$ in which all diagonal blocks are the identity, and let $\mathcal{Q}' = \mathcal{Q}/N_{\mathcal{Q}}$. (We may consider $\mathcal{Q}'$ to be the subgroup of $\mathcal{Q}$ in which all off-diagonal blocks are 0). Let π' denote the natural map $\mathcal{Q} \to \mathcal{Q}'$.

Claim 4.10. *We may write*

$$A' = \Lambda A'',$$

where $\Lambda \in \mathcal{Q}'$ is the diagonal matrix which is scaling by $e^{-\lambda_i t(c)}$ on $L_i(c)$, $A'' \in \mathcal{Q}$ and $\|A''\| = O(e^{\epsilon t})$.

Proof of claim. Suppose $x \in E^+[c]$ and $t = -t(c) < 0$ where $c \in C_1$ and $t(c)$ is as in Proposition 3.6. By construction, $-t > T_0(c)$, where $T_0(c)$ is as in Lemma 4.9. Then, the claim follows from (4.25) and Lemma 4.9 (d). □

Interpolation. We may write $A'' = DA_1$, where D is diagonal, and $\det A_1 = 1$. In view of Claim 4.10, $\|D\| = O(e^{\epsilon t})$ and $\|A_1\| = O(e^{\epsilon t})$.

We now connect A_1/K_c to the identity by the shortest possible path $\Gamma : [-t(c), 0] \to \mathcal{Q}K_c/K_c$, which stays in the subset $\mathcal{Q}K_c/K_c$ of the symmetric space $SL(V)/K_c$. (We parametrize the path so it has constant speed). This path has length $O(\epsilon t)$ where the implied constant depends only on the symmetric space.

Now for $-t(c) \leq t \leq 0$, let

$$A(t) = (\Lambda D)^{-t/t(c)} \Gamma(t). \tag{4.29}$$

Then $A(0)$ is the identity map, and $A(-t(c)) = A'$. Then, we define, for $x \in E^+[c]$ and $-t(c) \leq t \leq 0$,

$$\langle (g_t)_* u, (g_t)_* v \rangle_{g_t x} = \langle A(t)u, A(t)v \rangle_x.$$

We often omit the subscript from $\langle \cdot, \cdot \rangle_x$ and from the associated norm $\| \cdot \|_x$.

The subspace $L'_{ij}(x)$. Let $L'_{ij}(x)$ denote the orthogonal complement, relative to the inner product $\langle \cdot, \cdot \rangle_x$ of $L_{i,j-1}(x)$ in $L_{ij}(x)$.

Proposition 4.11. *The inner product $\langle \cdot, \cdot \rangle_x$ has the following properties:*

(a) *For each $x \in X$, the distinct eigenspaces $L_i(x)$ are orthogonal.*

(b) *For $v \in L'_{ij}(x) \subset H^{(++)}_{big}(x)$,*

$$(g_t)_* v = e^{\lambda_{ij}(x,t)} v' + v'', \tag{4.30}$$

where $v' \in L'_{ij}(g_t x)$, $v'' \in L_{i,j-1}(g_t x)$, and $\|v'\| = \|v\|$. Hence (since v' and v'' are orthogonal),

$$\|(g_t)_* v\| \geq e^{\lambda_{ij}(x,t)} \|v\|. \tag{4.31}$$

(c) *There exists a constant $\kappa > 1$ such that for a.e. $x \in X$ and for all $t > 0$,*

$$\kappa^{-1} t \le \lambda_{ij}(x,t) \le \kappa t.$$

(d) *There exists a constant $\kappa > 1$ such that for a.e $x \in X$ and for all $v \in H^{(++)}_{big}(x)$, and all $t \ge 0$,*

$$e^{\kappa^{-1} t}\|v\| \le \|(g_t)_* v\| \le e^{\kappa t}\|v\|.$$

(e) *Suppose $ij \in \Lambda''$. Then, for $y \in \mathfrak{B}[x] = W^+[x] \cap J[x]$ and $t \le 0$,*

$$\lambda_{ij}(x,t) = \lambda_{ij}(y,t).$$

Proof. Suppose first that $x = c$, where c and $E^+[c]$ are as in §3. Then, by construction, (a) and (b) hold. Also, from the construction, it is clear that the inner product $\langle \cdot, \cdot \rangle_c$ induces the inner product $\langle \cdot, \cdot \rangle_{ij,c}$ on $L_{ij}(c)/L_{i,j-1}(c)$.

Now by Proposition 4.4, for $x \in E^+[c]$, $P^+(x,c)L_{ij}(x) = L_{ij}(c)$, and for $\bar{u}, \bar{v} \in L_{ij}(x)/L_{i,j-1}(x)$, $\langle u, v\rangle_{ij,x} = \langle P^+(x,c)u, P^+(x,c)v\rangle_{ij,c}$. Therefore, (a), (b) and (e) hold for $x \in E^+[c]$, and also for $x \in E^+[c]$, the inner product $\langle \cdot, \cdot \rangle_x$ induces the inner product $\langle \cdot, \cdot \rangle_{ij,x}$ on $L_{ij}(x)/L_{i,j-1}(x)$. Now, (a),(b) and (e) hold for arbitrary $x \in J[c]$ since $A(t) \in \mathcal{Q}$.

Let $\psi_{ij} : \mathcal{Q}' \to \mathbb{R}_+$ denote the homomorphism taking the block-conformal matrix $\mathcal{Q}'$ to the scaling part of block corresponding to $L_{ij}/L_{i,j-1}$. Let $\varphi_{ij} = \psi_{ij} \circ \pi'$; then $\varphi_{ij} : \mathcal{Q} \to \mathbb{R}_+$ is a homomorphism.

From (4.29), we have, for $x \in E^+[c]$ and $-t(c) \le t \le 0$,

$$\log \varphi_{ij}(A(t)) = -t\lambda_i + \gamma(t),$$

where $-t\lambda_i$ is the contribution of $\Lambda^{-t/t(c)}$ and $\gamma(t)$ is the contribution of $D^{-t/t(c)}\Gamma(t)$. By Lemma 4.9 (d), we have for some $\kappa_1 > 1$

$$\kappa_1^{-1} < \lambda_i < \kappa_1 \tag{4.32}$$

and by Claim 4.10, for all $-t(c) \le t \le 0$,

$$|\gamma'(t)| = O(\epsilon) \tag{4.33}$$

where $\epsilon > 0$ is as in Claim 4.10, and the implied constant depends only on the symmetric space. Therefore (c) holds.

The lower bound in (d) now follows immediately from (b) and (c). The upper bound in (d) follows from (4.32) and (4.33). $\square$

Lemma 4.12. *For every $\delta > 0$ there exists a compact subset $K(\delta)$ with $\nu(K(\delta)) > 1 - \delta$ and a number $C_1(\delta) < \infty$ such for all $x \in K(\delta)$ and all v on $H^{(++)}_{big}(x)$ or $H^{(--)}_{big}(x)$,*

$$C_1(\delta)^{-1} \le \frac{\|v\|_x}{\|v\|_{x,H}} \le C_1(\delta),$$

where $\|\cdot\|$ is the dynamical norm defined in this subsection and $\|\cdot\|_{Y,x}$ is the AGY norm.

Proof. Since any two norms on a finite dimensional vector space are equivalent, there exists a function $\Xi_0 : X \to \mathbb{R}^+$ finite a.e. such that for all $x \in X$ and all $v \in H_{big}^{(++)}(x)$,

$$\Xi_0(x)^{-1}\|v\|_{Y,x} \leq \|v\|_x \leq \Xi_0(x)\|v\|_{Y,x}.$$

Since $\bigcup_{N \in \mathbb{N}}\{x \ : \ \Xi(x) < N\}$ is conull in X, we can choose $K(\delta) \subset X$ and $C_1 = C_1(\delta)$ so that $\Xi_0(x) < C_1(\delta)$ for $x \in K(\delta)$ and $\nu(K(\delta)) \geq (1-\delta)$. $\square$

5. Conditional Measure Lemmas

Motivation. We use notation from §2.3. For two (generalized) subspaces $\mathcal{U}'$ and $\mathcal{U}''$, $hd_x(\mathcal{U}',\mathcal{U}'')$ denote the Hausdorff distance between $\mathcal{U}' \cap B(x,1)$ and $\mathcal{U}'' \cap B(x,1)$. We can write

$$hd_{q_2}(U^+[q_2'], U^+[q_2]) = Q_t(q'-q),$$

where $Q_t : \mathcal{L}^-(q) \to \mathbb{R}$ is a map depending on q, u, ℓ, and t. The map Q_t is essentially the composition of flowing forward for time ℓ, shifting by $u \in U^+$ and then flowing forward again for time t. We then adjust t so that $hd_{q_2}(U^+[q_2'], U^+[q_2]) \approx \epsilon$, where $\epsilon > 0$ is a priori fixed.

In order to solve "technical difficulty #1" of §2.3, it is crucial to ensure that t does not depend on the precise choice of q' (it can depend on q, u, ℓ). The idea is to use the following trivial:

Lemma 5.1. *For any $\rho > 0$ there is a constant $c(\rho)$ with the following property: Let $A : \mathcal{V} \to \mathcal{W}$ be a linear map. Then there exists a proper subspace $\mathcal{M} \subset \mathcal{V}$ such that for any v with $\|v\| = 1$ and $d(v, \mathcal{M}) > \rho$, we have*

$$\|A\| \geq \|Av\| \geq c(\rho)\|A\|.$$

Proof of Lemma 5.1. The matrix A^tA is symmetric, so it has a complete orthogonal set of eigenspaces $W_1, \dots, W_m$ corresponding to eigenvalues $\mu_1 > \mu_2 > \dots \mu_m$. Let $\mathcal{M} = W_1^{\perp}$. $\square$

Now suppose the map $Q_t : \mathcal{L}^-(q) \to \mathbb{R}$ is of the form $Q_t(v) = \|\mathcal{Q}_t(v)\|$ where $\mathcal{Q}_t : \mathcal{L}^-(q) \to \mathbf{H}(q_2)$ is a linear map, and $\mathbf{H}(q_2)$ a vector space. This in fact happens in the first step of the induction where U^+ is the unipotent N (and we can take $\mathbf{H}(q_2) = W^+(q_2)/N$). We can then choose t, depending only on q, u and ℓ, such that the operator norm

$$\|\mathcal{Q}_t\| \equiv \sup_{v \in \mathcal{L}^-(q)} \frac{\|\mathcal{Q}_t(v)\|}{\|v\|} = \epsilon.$$

Then, we need to prove that we can choose $q' \in \mathcal{L}^-[q]$ such that $\|q'-q\| \approx 1$, q' avoids an a priori given set of small measure, and also $q'-q$ is at least ρ away from the "bad subspace" $\mathcal{M} = \mathcal{M}_u(q,\ell)$ of Lemma 5.1. (Actually, since we do not want the choice of q' to depend on the choice of u, we want to choose q' such that $q'-q$

avoids most of the subspaces $\mathcal{M}_u$ as $u \in U^+$ varies over a unit box). Then, for most u,

$$c(\rho)\epsilon \le \|\mathcal{Q}_t(q_2' - q_2)\| \le \epsilon,$$

and thus

$$c(\rho)\epsilon \le hd_{q_2}(U^+[q_2], U^+[q_2']) \le \epsilon, \tag{5.1}$$

as desired. In general we do not know that map $\mathcal{Q}_t$ is linear, because we do not know the dependence of the subspace $U^+(q)$ on q. To handle this problem, we can write

$$\mathcal{Q}_t(q' - q) = \mathcal{A}_t(F(q') - F(q))$$

where the map $\mathcal{A}_t : \mathcal{L}_{ext}(q)^{(r)} \to W^+(q_2)$ is linear (and can depend on q, u, ℓ), and the measurable map $F : \mathcal{L}^-[q] \to \mathcal{L}_{ext}[q]^{(r)}$ depends only on q. (See Lemma 6.7 below for a precise statement). The map F and the space $\mathcal{L}_{ext}[q]^{(r)}$ is defined in this section, and the linear map $\mathcal{A}_t = \mathcal{A}(q, u, \ell, t)$ is defined in §6.1.

We then proceed in the same way. We choose $t = \hat{\tau}(q, u, \ell, \epsilon)$ so that $\|\mathcal{A}_t\| = \epsilon$. (A crucial bilipshitz type property of the function $\hat{\tau}$ is proved in §7). In this section we prove Lemma 5.2, which roughly states that (for most q) we can choose $q' \in \mathcal{L}^-[q]$ while avoiding an a priori given set of small measure, so that $\|F(q') - F(q)\| \approx 1$ and also $F(q') - F(q)$ avoids most of a family of linear subspaces of $\mathcal{L}_{ext}[q]^{(r)}$ (which will be the "bad subspaces" of the linear maps $\mathcal{A}_t$ as u varies over U^+) Then as above, for most u, (5.1) holds.

The functions $P_{i,x}$ and $\mathfrak{P}$ and the space $\mathcal{L}_{ext}(x)$. Let the subspace $\mathcal{L}^-(x) \subset W^-(x)$ be the smallest such that the conditional measure $\nu_{W^-(x)}$ is supported on $\mathcal{L}^-[x]$. Write $\mathcal{L}^-(x) = (0,1) \otimes \mathcal{L}(x)$, where $\mathcal{L}(x) \subset H^1(M, \Sigma, \mathbb{R})$.

Since ν is invariant under N, the entropy of any $g_t \in A$ is positive. Therefore for ν-almost all $x \in X$, $\mathcal{L}(x) \neq \{0\}$.

As in §4.1, let $\mathcal{V}_i(x) \subset H^1(M, \Sigma, \mathbb{R})$ denote the subspace corresponding to the (cocycle) Lyapunov exponent λ_i. Let $\pi_i : \hat{V}_{k+1-i}(x) \to \hat{V}_{k+1-i}(x)/\hat{V}_{k-i}(x)$ denote the natural projection.

For $x \in X$, let $P_{i,x} \in Hom(\hat{V}_{k+1-i}(x)/\hat{V}_{k-i}(x), H^1(M, \Sigma, \mathbb{R}))$ denote the unique linear map such that for $\bar{x} \in \hat{V}_{k+1-i}(x)/\hat{V}_{k-i}(x)$, $P_{i,x}(\bar{x}) \in \mathcal{V}_i(x)$ and $\pi_i(P_{i,x}(\bar{x})) = \bar{x}$. Note that the $P_{i,x}$ satisfy the following:

$$P_{i,g_t x} = g_t \circ P_{i,x} \circ g_t^{-1}, \tag{5.2}$$

and

$$P_{i,x}(\bar{u}) - P_{i,y}(\bar{u}) \in \hat{V}_{k-i}(x). \tag{5.3}$$

Example. The space $\hat{V}_{k+1}/\hat{V}_k$ is one dimensional, and corresponds to the Lyapunov exponent $\lambda_1 = 1$. If we identify it with $\mathbb{R}$ in the natural way then $P_{1,x} : \mathbb{R} \to H^1(\cdot, \cdot, \mathbb{R})$ is given by the formula

$$P_{1,x}(\xi) = (\operatorname{Im} x)\xi \tag{5.4}$$

where for $x = (M, \omega)$, we write $\operatorname{Im} x$ for the imaginary part of ω.

Let

$$\mathfrak{P} : X \to \bigoplus_{i=1}^{k} Hom(\hat{V}_{k+1-i}(x)/\hat{V}_{k-i}(x), H^1(M, \Sigma, \mathbb{R}))$$

be given by

$$\mathfrak{P}(x) = (P_{1,x}, \ldots P_{k,x}),$$

We have

$$P^-(x, y) = \mathfrak{P}(y) \circ \mathfrak{P}(x)^{-1}, \tag{5.5}$$

where $P^-(x, y)$ is as in §4.2.

Let

$$\mathcal{L}_{ext}(x) \subset \bigoplus_{i=1}^{k} Hom(\hat{V}_{k+1-i}(x)/\hat{V}_{k-i}(x)), H^1(M, \Sigma, \mathbb{R}))$$

denote the smallest linear subspace which (up to measure 0) contains the linear span of the vectors $\{\mathfrak{P}(y) - \mathfrak{P}(x) \;\; : \;\; y \in W^-[x]\}$. We also set

$$\mathcal{L}_{ext}[x] = \mathfrak{P}(x) + \mathcal{L}_{ext}(x).$$

Note that for $y \in W^-[x]$, $\mathcal{L}_{ext}(y) = \mathcal{L}_{ext}(x)$, and (since $\mathfrak{P}(y) - \mathfrak{P}(x) \in \mathcal{L}_{ext}(x)$) $\mathcal{L}_{ext}[y] = \mathcal{L}_{ext}[x]$.

The space $\mathcal{L}_{ext}(x)^{(r)}$ and the function F. For a vector space V we use the notation $V^{\otimes m}$ to denote the m-fold tensor product of V with itself. If $f : V \to W$ is a linear map, we write $f^{\otimes m}$ for the induced linear map from $V^{\otimes m}$ to $W^{\otimes m}$. Let $j^{\otimes m} : V \to V^{\otimes m}$ denote the map $v \to v \otimes \ldots \otimes v$ (m-times).

Let $V^{\uplus m}$ denote $\bigoplus_{k=1}^{m} V^{\otimes m}$. If $f : V \to W$ is a linear map, we write $f^{\uplus m}$ for the induced linear map from $V^{\uplus m}$ to $W^{\uplus m}$ given by

$$f^{\uplus m}(v) = (f^{\otimes 1}(v), f^{\otimes 2}(v), \ldots f^{\otimes m}(v)).$$

Let r be an integer to be chosen later. Let $F : X \to \mathcal{L}_{ext}(x)^{\uplus r}$ denote the diagonal embedding

$$F(x) = \mathfrak{P}(x)^{\uplus r}.$$

Let

$$\mathcal{L}_{ext}(x)^{(r)} \subset \mathcal{L}_{ext}(x)^{\uplus r}$$

denote the smallest linear subspace which (up to measure 0) contains the linear span of the vectors $\{F(y) - F(x) \;\; : \;\; y \in W^-[x]\}$. We also set

$$\mathcal{L}_{ext}[x]^{(r)} = F(x) + \mathcal{L}_{ext}(x)^{(r)}.$$

Note that for $y \in W^-[x]$, $\mathcal{L}_{ext}[y]^{(r)} = \mathcal{L}_{ext}[x]^{(r)}$ and $\mathcal{L}_{ext}(y)^{(r)} = \mathcal{L}_{ext}(x)^{(r)}$.

In the rest of the paper, $\mathcal{B} \subset U^+$ is a "unit box". (The measure is assumed to be invariant under U^+).

To carry out the program outlined at the beginning of §5, we need the following:

Lemma 5.2. *For every $\delta > 0$ there exist constants $c_1(\delta) > 0$, $\epsilon_1(\delta) > 0$ with $c_1(\delta) \to 0$ and $\epsilon_1(\delta) \to 0$ as $\delta \to 0$, and also constants $\rho(\delta) > 0$, $\rho'(\delta) > 0$, and $C(\delta) < \infty$ such that the following holds:*

For any subset $K' \subset X$ with $\nu(K') > 1 - \delta$, there exists a subset K with $\nu(K) > 1 - c_1(\delta)$ such that the following holds: suppose for each $x \in X$ we have a measurable map from $\mathcal{B} \subset U^+(x)$ to proper subspaces of $\mathcal{L}_{ext}(x)^{(r)}$, written as $u \to \mathcal{M}_u(x)$, where $\mathcal{M}_u(x)$ is a proper subspace of $\mathcal{L}_{ext}(x)^{(r)}$. Then, for any $q \in K$ there exists $q' \in K'$ with

$$\rho'(\delta) \leq d(q, q') \leq 1 \tag{5.6}$$

and

$$\rho(\delta) \leq \|F(q') - F(q)\|_Y \leq C(\delta) \tag{5.7}$$

and so that

$$d(F(q') - F(q), \mathcal{M}_u(q)) > \rho(\delta) \qquad \text{for at least } (1 - \epsilon_1(\delta))\text{-fraction of } u \in \mathcal{B}. \tag{5.8}$$

This lemma is proved in the next subsection. The proof uses almost nothing about the maps F or the measure ν, other then the definition of $\mathcal{L}_{ext}(x)$. It may be skipped on first reading.

5.1*. **Proof of Lemma 5.2.**

The measure $\tilde{\nu}_x$. Let $\tilde{\nu}_x = F_* \left(\nu_{W^-(x)}\right)$ denote the pushforward of ν_{W^-} under F. Then $\tilde{\nu}_x$ is a measure supported on $\mathcal{L}_{ext}[x]^{(r)}$. (Note that for $y \in W^-[x]$, $\tilde{\nu}_x = \tilde{\nu}_y$).

Lemma 5.3. *For ν-almost all $x \in X$, for any $\epsilon > 0$ (which is allowed to depend on x), the restriction of the measure $\tilde{\nu}_x$ to the ball $B(F(x), \epsilon) \subset \mathcal{L}_{ext}[x]^{(r)}$ is not supported on a finite union of proper affine subspaces of $\mathcal{L}_{ext}[x]^{(r)}$.*

Outline of proof. Suppose not. Let $N(x)$ be the minimal integer N such that for some $\epsilon = \epsilon(x) > 0$, the restriction of $\tilde{\nu}_x$ to $B(F(x), \epsilon)$ is supported on N affine subspaces. Note that in view of (5.2) and (5.3), the induced action on $\mathcal{L}_{ext}$ (and hence on $\mathcal{L}_{ext}^{(r)}$) of g_{-t} for $t \geq 0$ is expanding. Then $N(x)$ is invariant under g_{-t}, $t \geq 0$. This implies that $N(x)$ is constant for ν-almost all x, and also that the only affine subspaces of $\mathcal{L}_{ext}[x]^{(r)}$ which contribute to $N(\cdot)$ pass through $F(x)$. Then, $N(x) > 1$ almost everywhere is impossible. Indeed, suppose $N(x) = k$ a.e., then pick y near x such that $F(y)$ is in one of the affine subspaces through $F(x)$; then there must be exactly k affine subspaces of non-zero measure passing though $F(y)$, but then at most one of them passes through $F(x)$. Thus, the measure restricted to a neighborhood of $F(x)$ gives positive weight to at least $k+1$ subspaces, contradicting our assumption. Thus, we must have $N(x) = 1$ almost everywhere; but then (after flowing by g_{-t} for sufficiently large $t > 0$) we see that for almost all x, $\tilde{\nu}_x$ is supported on a proper subspace of $\mathcal{L}_{ext}[x]^{(r)}$ passing through x, which contradicts the definition of $\mathcal{L}_{ext}(x)^{(r)}$. $\square$

Remark. Besides Lemma 5.3, the rest of the proof of Lemma 5.2 uses only the measurability of the map F.

The measure $\hat{\nu}_x$. Let $\tilde{W}^-[x]$ denote the intersection of the fundamental domain with $W^-[x]$. We choose finitely many relatively open disjoint sets $B_i \subset \tilde{W}^-[x]$ each of diameter at most 1, so that their union is conull in $\tilde{W}^-[x]$. Let $B[x]$ denote the set B_i containing x.

Let $\hat{\nu}_x = F_*(\nu_{W^-(x)}|_{B[x]})$, i.e. $\hat{\nu}_x$ is the pushforward under F of the restriction of $\nu_{W^-(x)}$ to $B[x]$. Then, for $y \in B[x]$, $\hat{\nu}_x = \hat{\nu}_y$. Suppose $\delta > 0$ is given. Since

$$\lim_{C\to\infty} \hat{\nu}_x(B(F(x), C)) = \hat{\nu}_x(\mathcal{L}_{ext}[x]^{(r)}),$$

there exists a function $c(x) > 0$ finite almost everywhere such that for almost all x,

$$\hat{\nu}_x(B(F(x), c(x))) > (1-\delta^{1/2})\hat{\nu}_x(\mathcal{L}_{ext}[x]^{(r)})$$

Therefore, we can find $C = C(\delta) > 0$ and a compact set K'_δ with $\nu(K'_\delta) > 1 - \delta^{1/2}$ such that for each $x \in K'_\delta$,

$$\hat{\nu}_x(B(F(x), C)) > (1-\delta^{1/2})\hat{\nu}_x(\mathcal{L}_{ext}[x]^{(r)}) \quad \text{for all } x \in K'_\delta. \tag{5.9}$$

In the rest of §5.1*, C will refer to the constant of (5.9).

Lemma 5.4. *For every $\eta > 0$ and every $N > 0$ there exists $\beta_1 = \beta_1(\eta, N) > 0$, $\rho_1 = \rho_1(\eta, N) > 0$ and a compact subset $K_{\eta,N}$ of measure at least $1-\eta$ such that for all $x \in K_{\eta,N}$, and any proper subspaces $\mathcal{M}_1(x), \dots, \mathcal{M}_N(x) \subset \mathcal{L}_{ext}(x)^{(r)}$,*

$$\hat{\nu}_x\left(B(F(x), C) \setminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}_k(x), \rho_1)\right) \geq \beta_1 \hat{\nu}_x(B(F(x), C)). \tag{5.10}$$

Outline of Proof. By Lemma 5.3, there exist $\beta_x = \beta_x(N) > 0$ and $\rho_x = \rho_x(N) > 0$ such that for any subspaces $\mathcal{M}_1(x), \dots \mathcal{M}_N(x) \subset \mathcal{L}_{ext}(x)^{(r)}$,

$$\hat{\nu}_x\left(B(F(x), C) \setminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}(x), \rho_x)\right) \geq \beta_x \hat{\nu}_x(B(F(x), C)). \tag{5.11}$$

Let $E(\rho_1, \beta_1)$ be the set of x such that (5.10) holds. By (5.11),

$$\nu\left(\bigcup_{\substack{\rho_1>0\\ \beta_1>0}} E(\rho, \beta)\right) = 1.$$

Therefore, we can choose $\rho_1 > 0$ and $\beta_1 > 0$ such that $\nu(E(\rho_1, \beta_1)) > 1 - \eta$. $\square$

Lemma 5.5. *For every $\eta > 0$ and every $\epsilon_1 > 0$ there exists $\beta = \beta(\eta, \epsilon_1) > 0$, a compact set $K_\eta = K_\eta(\epsilon_1)$ of measure at least $1-\eta$, and $\rho = \rho(\eta, \epsilon_1) > 0$ such that the following holds: Suppose for each $u \in \mathcal{B}$ let $\mathcal{M}_u(x)$ be a proper subspace of $\mathcal{L}_{ext}(x)^{(r)}$. Let*

$$E_{good}[x] = \{v \in B(F(x), C) \ : \ \text{for at least } (1-\epsilon_1)\text{-fraction of } u \text{ in } \mathcal{B}, \quad d(v - F(x), \mathcal{M}_u(x)) > \rho/2\}.$$

Then, for $x \in K_\eta$,

$$\hat{\nu}_x(E_{good}[x]) \geq \beta \hat{\nu}_x(B(F(x), C)). \tag{5.12}$$

Proof. Let $n = \dim \mathcal{L}_{ext}[x]^{(r)}$. By considering determinants, it is easy to show that there exists a constant $c_n = c_n(C) > 0$ depending on n and C such that if $v_1, \dots, v_n$ are points in a ball of radius C such that v_n is not within η of the subspace spanned by $v_1, \dots, v_{n-1}$, then $v_1, \dots, v_n$ are not within $c_n \eta$ of any $n-1$ dimensional subspace. Let $k_{max} \in \mathbb{N}$ denote the smallest integer greater then $1 + 2(n-1)/\epsilon_1$, and let $N = N(\epsilon_1) = \binom{k_{max}}{n-1}$. Let β_1, ρ_1 and $K_{\eta,N}$ be as in Lemma 5.4. Let $\beta = \beta(\eta, \epsilon_1) = \beta_1(\eta, N(\epsilon_1))$, $\rho = \rho(\eta, \epsilon_1) = \rho_1(\eta, N(\epsilon_1))/c_n$, $K_\eta(\epsilon_1) = K_{\eta, N(\epsilon_1)}$. Let $E_{bad}(x) = B(F(x), C) \setminus E_{good}(x)$. To simplify notation, we choose coordinates so that $F(x) = 0$. We claim that $E_{bad}(x)$ is contained in the union of the ρ_1-neighborhoods of at most N subspaces. Suppose this is not true. Then, for $1 \leq k \leq k_{max}$ we can inductively pick points $v_1, \dots, v_k \in E_{bad}(x)$ such that v_j is not within ρ_1 of any of the subspaces spanned by $v_{i_1}, \dots, v_{i_{n-1}}$ where $i_1 \leq \dots \leq i_n < j$. Then, any $n-1$-tuple of points $v_{i_1}, \dots, v_{i_{n-1}}$ is not contained within $\rho = c_n \rho_1$ of a single subspace. Now, since $v_i \in E_{bad}(x)$, there exists $U_i \in \mathcal{B}$ with $|U_i| \geq \epsilon_1$ such that for all $u \in U_i$, $d(v_i, M_u) < \rho/2$. We now claim that for any $1 \leq i_1 < i_2 < \dots < i_{n-1} \leq k$,

$$U_{i_1} \cap \dots U_{i_{n-1}} = \emptyset. \tag{5.13}$$

Indeed, suppose u belongs to the intersection. Then each of the $v_{i_1}, \dots v_{i_{n-1}}$ is within $\rho/2$ of the single subspace $\mathcal{M}_u$, but this contradicts the choice of the v_i. This proves (5.13). Now,

$$\epsilon_1 k_{max} \leq \sum_{i=1}^{k_{max}} |U_i| \leq (n-1) \left| \bigcup_{i=1}^{k_{max}} U_i \right| \leq (n-1)|\mathcal{B}| \leq 2(n-1).$$

This is a contradiction, since $k_{max} > 1 + 2(n-1)/\epsilon_1$. This proves the claim. Now (5.10) implies that

$$\hat{\nu}_x(E_{good}(x)) \geq \hat{\nu}_x(B(F(x), C) \setminus \bigcup_{k=1}^{N} \operatorname{Nbhd}(\mathcal{M}_k(x), \rho)) \geq \beta \hat{\nu}_x(B(F(x), C)).$$

$\square$

Proof of Lemma 5.2. Let

$$K'' = \{x \in X \;\;:\;\; \nu_{W^-(x)}(K' \cap B[x]) \geq (1-\delta^{1/2})\nu_{W^-(x)}(B[x]).$$

Thus, for $x \in K''$,

$$\hat{\nu}_x(F(K'')) \geq (1-\delta^{1/2})\hat{\nu}_x(\mathcal{L}_{ext}[x]^{(r)}). \tag{5.14}$$

Then, for almost all x, $\nu_{W^-(x)}((K'')^c) \leq \nu_{W^-(x)}((K')^c)\delta^{-1/2}$, hence $\nu((K'')^c) \leq \delta^{-1/2}\nu((K')^c) = \delta^{1/2}$. Hence, $\nu(K'') \geq 1-\delta^{1/2}$.

Let $\beta(\eta,\epsilon_1)$ be as in Lemma 5.5. Let

$$c(\delta) = \min(\delta, \inf\{(\eta^2+\epsilon_1^2)^{1/2} \;\;:\;\; \beta(\eta,\epsilon_1) \geq 8\delta^{1/2}\}).$$

We have $c(\delta) \to 0$ as $\delta \to 0$. By the definition of $c(\delta)$ we can choose $\eta = \eta(\delta) < c(\delta)$ and $\epsilon_1 = \epsilon_1(\delta) < c(\delta)$ so that $\beta(\eta,\epsilon_1) \geq 8\delta^{1/2}$.

Now suppose $x \in K'' \cap K'_\delta$. Then, by (5.9) and (5.14),

$$\hat{\nu}_x(F(K') \cap B(F(x),C)) \geq (1-4\delta^{1/2})\hat{\nu}_x(B(F(x),C)). \tag{5.15}$$

By (5.12), for $x \in K_\eta$,

$$\hat{\nu}_x(E_{good}(x)) \geq 8\delta^{1/2}\hat{\nu}_x(B(F(x),C)). \tag{5.16}$$

Let $K = K'' \cap K'_\delta \cap K_\eta$. We have $\nu(K) \geq 1 - 2\delta^{1/2} - c(\delta)$, so $\nu(K) \to 1$ as $\delta \to 0$. Also, if $q \in K$, by (5.15) and (5.16),

$$F(K') \cap E_{good}(q) \cap B(F(x),C) \neq \emptyset.$$

Thus, we can choose $q' \in B[q] \cap F(K')$ such that $F(q') \in E_{good}[q] \cap B(F(q),C))$. Then (5.8) holds with $\rho = \rho(\eta(\delta),\epsilon_1(\delta)) > 0$. Also the upper bound in (5.6) holds since $B[x]$ has diameter at most 1, and the upper bound in (5.7) holds since $F(q') \in B(F(q),C)$. Since all $\mathcal{M}_u(q)$ contain the origin q, the lower bound in (5.7) follows from (5.8). Finally, the lower bound in (5.6) follows from combining the lower bound in (5.7) with (5.4). $\square$

6. Divergence of Generalized Subspaces.

The groups $\mathcal{G}$ and $\mathcal{G}_+$. Let $\mathcal{G}(x)$ denote the affine group of $W^+[x]$, i.e. the group of affine maps from $W^+[x]$ to itself. Let $Q(x)$ denote the group of linear maps from $W^+(x)$ to itself which preserve the flag $\{0\} \subset V_1(x) \subset \cdots \subset V_n(x) = W^+(x)$, and let $Q_+(x) \subset Q(x)$ denote the unipotent subgroup of maps which are the identity on $V_{i+1}(x)/V_i(x)$. Let $\mathcal{G}_+(x)$ denote the subgroup of $\mathcal{G}(x)$ of affine maps in which the linear part lies in $Q_+(x)$. Then, in view of Lemma 4.1, for $y \in W^+[x]$, $\mathcal{G}(y) = \mathcal{G}(x)$, $Q(y) = Q(x)$, $Q_+(y) = Q_+(x)$ and $\mathcal{G}_+(y) = \mathcal{G}_+(x)$. Also, $Q(x)$ is the stabilizer of x in $\mathcal{G}_+(x)$.

We denote by $\mathrm{Lie}(\mathcal{G}_+)(x)$ the Lie algebra of $\mathcal{G}_+(x)$, etc.

We will often identify $W^+(x)$ with the translational part of the Lie algebra $\mathrm{Lie}(\mathcal{G}_+)(x)$. Then, we have an exponential map $\exp : W^+(x) \to \mathcal{G}_+(x)$, taking $v \in W^+(x)$ to $\exp v \in \mathcal{G}_+(x)$. Then, $\exp v : W^+ \to W^+$ is translation by v.

Generalized subspaces. Let $U'(x) \subset \mathcal{G}_+(x)$ be a subgroup. We write

$$U'[x] = \{ux \ : \ u \in U'(x)\}$$

and call $U'[x]$ a generalized subspace. We have $U'[x] \subset W^+[x]$.

Standing Assumption. We are assuming that for almost every $x \in X$ there is a distinguished subgroup $U^+(x)$ of $\mathcal{G}_+(x)$ so that the conditional measure of ν along $U^+[x]$ is induced from the Haar measure on $U^+(x)$. We are also assuming that the foliation whose leaves are sets of the form $U^+[x]$ is invariant under the geodesic flow.

Warning. Suppose $u \in U^+(x)$. Then, we have $U^+[ux] = U^+[x]$. However, since we are identifying $W^+(x)$ and $W^+(ux)$ via a translation, and not via the action of $u \in \mathcal{G}_+$, it is not always true that $U^+(ux) = U^+(x)$. Instead we have

$$U^+(ux) = u_0 U(x) u_0^{-1}, \tag{6.1}$$

where $u_0 \in Q_+(x)$ is the linear part of $u \in \mathcal{G}_+(x)$. Thus,

$$\mathrm{Lie}(U^+)(ux) = u_0 \,\mathrm{Lie}(U^+)(x) u_0^{-1}.$$

Lyapunov subspaces. For a bundle W which is a quotient bundle of a subbundle of $H^{(++)}_{big}$, we let $\mathcal{V}_i(W)(x)$ denote the Lyapunov subspaces of g_t acting on W, and let $\lambda_i(W)$ denote the corresponding Lyapunov exponents. We always number the Lyapunov exponents so that $\lambda_1 > \lambda_2 \geq .... \geq \lambda_n$. Let $V_i(W) = \bigoplus_{j=1}^{i} \mathcal{V}_i(W)$.

If there is no potential for confusion about which bundle W is used, we write $\mathcal{V}_i(x)$, $V_i(x)$ and λ_i instead of $\mathcal{V}_i(W)(x)$, $V_i(W)(x)$ and $\lambda_i(W)$.

Since $\mathrm{Lie}(U^+)(x)$ and $\mathrm{Lie}(Q)(x)$ are equivariant under the g_t action, we have

$$\mathrm{Lie}(U^+)(x) = \bigoplus_i \mathcal{V}_i(\mathrm{Lie}(U^+))(x), \qquad \mathrm{Lie}(Q)(x) = \bigoplus_i \mathcal{V}_i(\mathrm{Lie}(Q))(x).$$

The spaces $\mathcal{H}(x)$ and $\mathcal{H}_+(x)$. Let $\mathcal{H}(x) = \mathrm{Hom}(\mathrm{Lie}(U^+)(x), \mathrm{Lie}(\mathcal{G}_+)(x))$.

For every $M \in \mathcal{H}(x)$, we can write

$$M = \sum_{ij} M_{ij} \quad \text{where } M_{ij} \in \mathrm{Hom}(\mathcal{V}_j(\mathrm{Lie}(U^+))(x), \mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_+))(x)). \tag{6.2}$$

Let

$$\mathcal{H}_+(x) = \{M \in \mathcal{H}(x) \ : \ M_{ij} = 0 \text{ if } \lambda_i \leq \lambda_j\}.$$

Then, $\mathcal{H}_+$ is the direct sum of all the positive Lyapunov subspaces of the action of g_t on $\mathcal{H}$.

Parametrization of generalized subspaces. Suppose $M \in \mathcal{H}(x)$ is such that $(I+M)\operatorname{Lie}(U^+)(x)$ is a subalgebra of $\operatorname{Lie}(\mathcal{G}_+)(x)$. We say that the pair $(M,v) \in \mathcal{H}(x) \times W^+(x)$ parametrizes the generalized subspace $\mathcal{U}$ if

$$\mathcal{U} = \{\exp[(I+M)u]\,(x+v) \ : \ u \in \operatorname{Lie}(U^+)(x)\}.$$

(Thus, $\mathcal{U}$ is the orbit of the subgroup $\exp[(I+M)\operatorname{Lie}(U^+)(x)]$ through the point $x+v \in W^+(x)$.) In this case we write $\mathcal{U} = \mathcal{U}(M,v)$.

Example 1. We give an example of a non-linear generalized subspace. Suppose for simplicity that W^+ has two Lyapunov exponents λ_1 and λ_2 with $\lambda_1 = 2\lambda_2$. Let $e_1(x)$ be $e_2(x)$ be unit vectors so that $\mathcal{V}_1(W^+)(x) = e_1(x)$, and $\mathcal{V}_2(W^+)(x) = e_2(x)$.

Let $i : W^+(x) \to \mathbb{R}^3$ be the map sending $x + ae_1(x) + be_2(x) \to (a,b,1) \in \mathbb{R}^3$. We identify $W^+(x)$ with its image in $\mathbb{R}^3$ under i. Then, we can identify

$$\mathcal{G}_+(x) = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \qquad \operatorname{Lie}(\mathcal{G}_+(x)) = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose

$$U^+(x) = \left\{ \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \ : \ t \in \mathbb{R} \right\}, \qquad \operatorname{Lie}(U^+(x)) = \left\{ \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \ : \ t \in \mathbb{R} \right\}.$$

Then, $U^+[x]$ is the parabola $\{x + te_1(x) + \frac{t^2}{2}e_2(x) \ : \ t \in \mathbb{R}\} \subset W^+[x]$.

Transversals. Note that we have

$$\operatorname{Lie}(\mathcal{G}_+)(x) = \operatorname{Lie}(Q_+)(x) \oplus W^+(x)$$

where we identify $W^+(x)$ with the subspace of $\operatorname{Lie}(\mathcal{G}_+)(x)$ corresponding to pure translations.

For each i, and each $x \in X$, let $Z_{i1}(x) \subset \mathcal{V}_i(W^+)(x) \subset \mathcal{V}_i(\operatorname{Lie}(\mathcal{G}_+))(x)$ be a linear subspace so that

$$\mathcal{V}_i(\operatorname{Lie}(\mathcal{G}_+))(x) = Z_{i1}(x) \oplus \mathcal{V}_i(\operatorname{Lie}(U^+) + \operatorname{Lie}(Q))(x).$$

Let $Z_{i2}(x) \subset \mathcal{V}_i(\operatorname{Lie}(Q))(x)$ be such that

$$\mathcal{V}_i(\operatorname{Lie}(U^+) + \operatorname{Lie}(Q))(x) = \mathcal{V}_i(\operatorname{Lie}(U^+))(x) \oplus Z_{i2}(x).$$

Let $Z_i(x) = Z_{i1}(x) \oplus Z_{i2}(x)$, and let $Z(x) = \bigoplus_i Z_i(x)$. We say that $Z(x) \subset \operatorname{Lie}(\mathcal{G}_+)(x)$ is an *Lyapunov-admissible transversal* to $\operatorname{Lie}(U^+)(x)$.

Note that $Z_{i1}(x) = Z(x) \cap \mathcal{V}_i(W^+)(x)$.

Example 2. Suppose $U^+(x)$ is as in Example 1. Then,

$$\mathcal{V}_2(\operatorname{Lie}(\mathcal{G}_+))(x) = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{V}_2(\operatorname{Lie}(U^+))(x) = \left\{ \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \ : \ t \in \mathbb{R} \right\},$$

$$\mathcal{V}_2(\mathrm{Lie}(Q_+))(x) = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{V}_1(\mathrm{Lie}(\mathcal{G}_+)) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\mathcal{V}_1(\mathrm{Lie}(U^+)) = \mathcal{V}_1(\mathrm{Lie}(Q)) = \{0\}$. Therefore, $Z_{21}(x) = \{0\}$, and

$$Z_{22}(x) = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Z_{11}(x) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Z_{21}(x) = \{0\}.$$

We note that in this example, the transveral Z was uniquely determined (and is in fact invariant under the flow g_t). This is a consequence of the fact that we chose an example with simple Lyapunov spectrum, and would not be true in general.

Parametrization adapted to a transversal. We say that the parametrization $(M, v) \in \mathcal{H}(x) \times W^+(x)$ of a generalized subspace $\mathcal{U} = \mathcal{U}(M, v)$ is adapted to the transversal $Z(x)$ if

$$v \in Z(x) \cap W^+(x)$$

and

$$Mu \in Z(x) \quad \text{ for all } u \in \mathrm{Lie}(U^+)(x).$$

Lemma 6.1. *Suppose the pair $(M', v') \in \mathcal{H}_+(x) \times W^+(x)$ parametrizes a generalized subspace $\mathcal{U}$. Let $Z(x)$ be a Lyapunov-admissible transversal. Then, there exists a pair (M, v) which parametrizes $\mathcal{U}$ and is adapted to $Z(x)$. If we write*

$$M' = \sum_{ij} M'_{ij}$$

as in (6.2), and

$$v' = \sum_j v'_j,$$

where $v'_j \in \mathcal{V}_j(x)$, then $M = \sum_{ij} M_{ij}$ and $v = \sum_i v_i$ are given by formulas of the form

$$v_i = L_i v'_i + p_i(v', M') \tag{6.3}$$

$$M_{ij} = L_{ij} M'_{ij} + p_{ij}(v', M') \tag{6.4}$$

where L_i is a linear map and p_i is a polynomial in the v'_j and M'_{jk} which depends only on the v'_j with $\lambda_j < \lambda_i$ and the M'_{jk} with $\lambda_j - \lambda_k < \lambda_i$. Similarly, L_{ij} is a linear map, and p_{ij} is a polynomial which depends on the v'_j and the M'_{kl} with $\lambda_k - \lambda_l < \lambda_i - \lambda_j$.

If we assume in addition that (M', v') is adopted to another Lyapunov-admissible transversal $Z'(x)$, then L_i and L_{ij} can be taken to be invertible linear maps (depending only on $Z(x)$ and $Z'(x)$).

Proof. We can choose a subspace $T(x) \subset \mathrm{Lie}(U^+)(x)$, so that

$$\mathrm{Lie}(U^+)(x) + \mathrm{Lie}(Q)(x) = T(x) \oplus \mathrm{Lie}(Q)(x).$$

(In particular, if $\mathrm{Lie}(U^+)(x) \cap \mathrm{Lie}(Q) = \{0\}$, $T(x) = \mathrm{Lie}(U^+)(x)$.) Then,

$$\mathrm{Lie}(\mathcal{G}_+)(x) = (Z(x) \cap W^+(x)) \oplus T(x) \oplus \mathrm{Lie}(Q)(x).$$

Thus, for any vector $Y \in \mathrm{Lie}(\mathcal{G}_+)(x)$, we can write

$$Y = \pi_Q(Y) + \pi_Z(Y) + \pi_T(Y), \tag{6.5}$$

where $\pi_Q(Y) \in \mathrm{Lie}(Q)(x)$, $\pi_Z(Y) \in Z(x) \cap W^+(x)$, $\pi_T(Y) \in T(x)$.

Let $\tilde{u} \in T(x)$ be such that (in $W^+(x)$)

$$x + v \equiv \exp[(I + M')\tilde{u}](x + v') \in x + Z(x) \cap W^+(x). \tag{6.6}$$

Then there exists $q \in \mathrm{Lie}(Q)(x)$, $z \in Z(x) \cap W^+(x)$ such that in $\mathcal{G}_+(x)$,

$$\exp[(I + M')\tilde{u}] \exp(v') = \exp(z) \exp(q). \tag{6.7}$$

Write $\tilde{u} = \sum_i \tilde{u}_i$, where $\tilde{u}_i \in \mathcal{V}_i(\mathrm{Lie}(U^+))(x)$. Also, write $q = \sum_i q_i$, where $q_i \in \mathcal{V}_i(\mathrm{Lie}(Q))(x)$. We now plug in to (6.7), and compare terms in $\mathcal{V}_i(\mathcal{G}_+)(x)$. (Here we work with matrix entries in the group, and not the algebra). We get equations of the form

$$\tilde{u}_i + v'_i + p_i = z_i + q_i,$$

where p_i is a polynomial in the $\tilde{u}_j$ and q_j for $\lambda_j < \lambda_i$, and in the M'_{jk} for $\lambda_j - \lambda_k < \lambda_i$. Then, the equation can be solved inductively, starting with the equation with i maximal (and thus λ_i minimal). We get,

$$\tilde{u}_i = -\pi_T(v'_i + p_i), \quad z_i = \pi_Z(v'_i + p_i), \quad q_i = \pi_Q(v'_i + p_i),$$

where π_Q, π_T and π_Z as as in (6.5). This shows that $v = \exp(z)v'$ has the form given in (6.3).

Let $U' = \exp((I + M')\,\mathrm{Lie}(U^+)(x))$. By our assumptions, U' is a subgroup of $\mathcal{G}_+$. Therefore, for $\tilde{u}$ as in (6.6),

$$\mathcal{U} = U'[x + v'] = U' \exp(-(I + M')\tilde{u})[x + v] = U'[x + v].$$

Hence, by (6.1), since the linear part of $\exp[(I + M')\tilde{u}]$ is $\exp q$,

$$U'(x + v) = U'(\exp[(I + M')\tilde{u}]x) = (\exp q)U'(x)(\exp q)^{-1}.$$

Let $M'' = (\exp q)M'(\exp q)^{-1}$. (Note that M''_{ij} depends only v' and on the M'_{kl} with $\lambda_k - \lambda_l \leq \lambda_i - \lambda_j$.) Then, (M'', v) is also a parametrization of $\mathcal{U}$. To make M'' adapted to $Z(x)$ we proceed as follows:

For $u \in \mathrm{Lie}(\mathcal{G}_+)(x)$, we can write $u = u'' + z''$, where $u'' \in \mathrm{Lie}(U^+)(x)$ and $z'' \in Z(x)$. Let $\pi^Z_{U^+} : \mathrm{Lie}(\mathcal{G}_+) \to \mathrm{Lie}(U^+)$ be the linear map sending u to u''. Let $u' \in \mathrm{Lie}(U^+)(x)$ be such that $u' + M''u' = u + z$, where $z \in Z$. Then,

$$u' + \pi^Z_{U^+}(M''u')) = u,$$

hence

$$u' = (I + \pi^Z_{U^+} \circ M'')^{-1} u.$$

Let

$$M = (I + M'')(I + \pi^Z_{U^+} \circ M'')^{-1} - I. \tag{6.8}$$

Then for all $u \in \mathrm{Lie}(U^+)(x)$, $Mu = (I+M)u - u = (I + M'')u' - u \in Z(x)$. Thus (M, v) is adapted to $Z(x)$. Since $M'' \in \mathcal{H}_+(x)$,

$$\pi^Z_{U^+} \circ M'' = \sum_{i<j} \pi^Z_{U^+} \circ M''_{ij},$$

where $M''_{ij} \in \mathrm{Hom}(\mathcal{V}_j(\mathrm{Lie}(U^+)), \mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_+))$. Since $Z(x)$ is a Lyapunov-admissible transversal, $\pi^Z_{U^+}$ takes $\mathcal{V}_i(\mathrm{Lie}(\mathcal{G}_+))$ to $\mathcal{V}_i(\mathrm{Lie}(\mathcal{U}^+))$. Therefore,

$$\pi^Z_{U^+} \circ M''_{ij} \in \mathrm{Hom}(\mathcal{V}_j(\mathrm{Lie}(U^+)), \mathcal{V}_i(\mathrm{Lie}(U^+)).$$

Thus, $\pi^Z_{U^+} \circ M''$ is nilpotent. Then (6.4) follows form (6.8). $\square$

The map S^Z_x. Suppose Z is a Lyapunov-admissible transversal to $U^+(x)$. Then, let $S^Z_x : \mathcal{H}_+(x) \times W^+(x) \to \mathcal{H}_+(x) \times W^+(x)$ be given by

$$S^Z_x(M', v') = (M, v)$$

where M' and v' are given by (6.4) and (6.3) respectively. Note that S^Z_x is a polynomial, but is *not* a linear map in the entries of M' and v'. To deal with the non-linearity, we work with certain tensor product spaces defined below.

Tensor Products: the spaces $\hat{\mathbf{H}}$, $\tilde{\mathbf{H}}$ and the maps $\mathbf{j}$. As in §5, for a vector space V and a map $f : V \to W$ we use the notations $V^{\otimes m}$, $V^{\uplus m}$, $f^{\otimes m}$, $f^{\uplus m}$, $j^{\otimes m}$, $j^{\uplus m}$.

Let m be the number of distinct Lyapunov exponents on $\mathcal{H}_+$, and let n be the number of distinct Lyapunov exponents on W^+. Let $(\alpha; \beta) = (\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n)$ be a multi-index, and let

$$\tilde{\mathbf{H}}^{(\alpha;\beta)}(x) = \bigotimes_{i=1}^m (\mathcal{H}_+(x) \cap \mathcal{V}_i(x))^{\otimes \alpha_i} \otimes \bigotimes_{j=1}^n (W^+(x) \cap \mathcal{V}_i(x))^{\otimes \beta_i}$$

and let

$$\hat{\mathbf{H}}(x)^{(\alpha;\beta)}(x) = \bigotimes_{i=1}^m \mathcal{H}_+(x)^{\otimes \alpha_i} \otimes \bigotimes_{j=1}^n W^+(x)^{\otimes \beta_i}.$$

We have a natural map $\hat{\pi}^{(\alpha;\beta)} : \hat{\mathbf{H}}^{(\alpha;\beta)}(x) \to \tilde{\mathbf{H}}^{(\alpha;\beta)}(x)$ given by

$$\begin{aligned} &\hat{\pi}^{(\alpha;\beta)}(Y_1 \otimes \dots \otimes Y_m \otimes (Y_1') \otimes \dots \otimes (Y_n')) = \\ &\qquad = \pi_1^{\otimes \alpha_1}(Y_1) \otimes \dots \otimes \pi_m^{\otimes \alpha_m}(Y_m) \otimes (\pi_1')^{\otimes \beta_1}(Y_1') \otimes \dots \otimes (\pi_n')^{\otimes \beta_n}(Y_n'), \end{aligned}$$

where $\pi_i : \mathcal{H}_+(x) \to \mathcal{V}_i(\mathcal{H}_+)(x)$ and $\pi'_j : W^+(x) \to \mathcal{V}_j(W^+)(x)$ are the natural projections associated to the direct sum decompositions $\mathcal{H}^+(x) = \bigoplus_{i=1}^m \mathcal{V}_i(H^+)(x)$ and $W^+(x) = \bigoplus_{j=1}^n \mathcal{V}_j(W^+)(x)$.

Let $\mathcal{S}$ be a finite collection of multi-indices (chosen in Lemma 6.2 below). Then, let

$$\tilde{\mathbf{H}}(x) = \bigoplus_{(\alpha;\beta)\in\mathcal{S}} \tilde{\mathbf{H}}^{(\alpha;\beta)}, \qquad \hat{\mathbf{H}}(x) = \bigoplus_{(\alpha;\beta)\in\mathcal{S}} \hat{\mathbf{H}}^{(\alpha;\beta)}$$

Let $\hat{\pi} : \hat{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(x)$ be the linear map with coincides with $\hat{\pi}^{(\alpha;\beta)}$ on each $\hat{\mathbf{H}}^{(\alpha;\beta)}$.

Let $\hat{\mathbf{j}}^{(\alpha;\beta)} : \mathcal{H}_+(x) \times W^+(x) \to \hat{\mathbf{H}}^{(\alpha;\beta)}(x)$ be the "diagonal embedding"

$$\hat{\mathbf{j}}^{(\alpha;\beta)}(M, v) = M \otimes M \ldots \otimes M \otimes v \otimes \ldots \otimes v,$$

and let $\hat{\mathbf{j}} : \mathcal{H}_+(x) \times W^+(x) \to \hat{\mathbf{H}}(x)$ be the linear map $\bigoplus_{(\alpha;\beta)\in\mathcal{S}} \hat{\mathbf{j}}^{(\alpha;\beta)}$. Let

$$\mathbf{j} : \mathcal{H}_+(x) \times W^+(x) \to \tilde{\mathbf{H}}(x)$$

denote $\hat{\pi} \circ \hat{\mathbf{j}}$. Note that the image of $\hat{\mathbf{j}}$ spans $\hat{\mathbf{H}}(x)$ and the image of $\mathbf{j}$ spans $\tilde{\mathbf{H}}(x)$.

Induced linear maps on $\hat{\mathbf{H}}(x)$ and $\tilde{\mathbf{H}}(x)$. Suppose $F_t : \mathcal{H}_+(x) \to \mathcal{H}_+(y)$ and $F'_t : W^+(x) \to W^+(y)$ are linear maps. Let $f_t = (F_t, F'_t)$. Then, f_t induces a linear map $\mathbf{f}_t : \hat{\mathbf{H}}(x) \to \hat{\mathbf{H}}(y)$. If F_t sends each $\mathcal{V}_i(\mathcal{H}_+)(x)$ to each $\mathcal{V}_i(\mathcal{H}_+)(y)$ and F'_t sends each $\mathcal{V}_j(W^+)(x)$ to $\mathcal{V}_j(W^+)(y)$, then f_t also induces a linear map $\mathbf{f}_t : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(y)$.

Note that $\tilde{\mathbf{H}}(x) \subset \hat{\mathbf{H}}(x) \subset H_{big}^{(++)}(x)$ where $H_{big}^{(++)}(x)$ is as in §3.

Notation. For a map $A : W^+(x) \to W^+(y)$, let $A_* : \mathrm{Lie}(\mathcal{G}_+)(x) \to \mathrm{Lie}(\mathcal{G}_+)(y)$ denote the map

$$A_*(Y) = A \circ Y_1 \circ A^{-1} + A \circ Y_2 \tag{6.9}$$

where for $Y \in \mathrm{Lie}(\mathcal{G}_+)(x)$, Y_1 is the linear part of Y and Y_2 is the pure translation part.

The map u_*. Suppose $u \in U^+(x)$. Let $u_* : \mathcal{H}_+(x) \times W^+(x) \to \mathcal{H}^+(ux) \times W^+(ux)$ denote the map

$$u_*(M, v) = (M \circ P^+(ux, x)_*, \exp((I + M)Y)(x + v) - \exp(Y)x), \tag{6.10}$$

where $Y = \log u$ and we are using (6.9) to define $P^+(ux, x)_*$. We claim that if $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$ parametrizes generalized subspace $\mathcal{U}$, then $u_*(M, v) \in \mathcal{H}_+(ux) \times W^+(ux)$ parametrizes the same subspace $\mathcal{U}$. Indeed, by Proposition 4.4,

$$P^+(ux, x)_* \,\mathrm{Lie}(U^+)(ux) = \mathrm{Lie}(U^+)(x),$$

and furthermore, $P^+(ux, x)_* \mathcal{V}_i(\mathrm{Lie}(U^+))(ux) = \mathcal{V}_i(\mathrm{Lie}(U^+))(x)$. Also,

$$\exp((I + M)Y)(x + v) \in \mathcal{U} = \mathcal{U}(M, v).$$

Therefore, since $\exp(Y)x = ux$,

$$ux + (\exp((I + M)Y)(x + v) - \exp(Y)x) \in \mathcal{U}.$$

Thus, $u_*(M, v) \in \mathcal{H}_+(ux) \times W^+(ux)$ as defined in (6.10) parametrizes the same generalized subspace $\mathcal{U}$ as $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$.

Lemma 6.2. *For an appropriate choice of $\mathcal{S}$, the following hold:*

(a) *Let $Z(x)$ be an Lyapunov-admissible transversal to $U^+(x)$. there exists a linear map $\mathbf{S}_x^{Z(x)} : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(x)$ such that for all $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$,*

$$(\mathbf{S}_x^{Z(x)} \circ \mathbf{j})(M, v) = (\mathbf{j} \circ S_x^{Z(x)})(M, v).$$

(b) *Suppose $u \in U^+(x)$, and let $Z(ux)$ be an admissible transversal to $U^+(ux)$. Then, there exists a linear map $(u)_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ such that for all $(M, v) \in \mathcal{H}^+(x) \times W^+(x)$,*

$$((u)_* \circ \mathbf{j})(M, v) = (\mathbf{j} \circ S_{ux}^{Z(ux)} \circ u_*)(M, v),$$

where $u_* : \mathcal{H}_+(x) \times W^+(x) \to \mathcal{H}^+(ux) \times W^+(ux)$ is as in (6.10).

Proof. Part (a) formally follows from the universal property of the tensor product and the partial ordering in (6.3) and (6.4). Instead of giving the notationally complex argument, we just present an example below. To prove (b), note that the equations coming from (6.10), have the same form as in (6.3) and (6.4). Thus, there exists a map $\tilde{u}_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ such that $\tilde{u}_* \circ \mathbf{j} = \mathbf{j} \circ u_*$, where u_* is as in (6.10). Now, we can define $(u)_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ to be $\mathbf{S}_{ux}^{Z(ux)} \circ \tilde{u}_*$, where $\mathbf{S}_{ux}^{Z(ux)}$ is as in (a). $\square$

Example 3. Suppose U^+ is as in Example 1 and Example 2. Let

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathcal{V}_2(\mathrm{Lie}(U^+))(x) = \mathbb{R}F$, $\mathcal{V}_1(\mathrm{Lie}(\mathcal{G}_+))(x) = \mathbb{R}E_1$. Then, for $M \in \mathcal{H}_+(x)$, the only non-zero component is $M_{12} \in \mathrm{Hom}(\mathcal{V}_2(\mathrm{Lie}(U^+))(x), \mathcal{V}_1(\mathrm{Lie}(\mathcal{G}_+))(x))$, which is 1-dimensional. Let

$$\Psi \in \mathrm{Hom}(\mathcal{V}_2(\mathrm{Lie}(U^+))(x), \mathcal{V}_1(\mathrm{Lie}(\mathcal{G}_+))(x))$$

denote the element such that $\Psi F = E_1$, so that $\mathcal{H}_+ = \mathbb{R}\Psi$.

With the choice of transversal Z given in Example 2, the equations (6.3) and (6.4) become:

$$v_1 = M'_{12} v'_2 + v'_1 + (v'_2)^2, \quad v_2 = 0, \quad M_{12} = M'_{12}. \tag{6.11}$$

Then we can choose $\mathcal{S} = \{(1; 0, 0), (0; 1, 0), (0; 0, 1), (1; 0, 1), (0; (1, 0)), (0; (0, 2)),$ so that (dropping the (x)),

$$\tilde{\mathbf{H}} = \mathcal{H}_+ \oplus \mathcal{V}_1(W^+) \oplus \mathcal{V}_2(W^+) \oplus (\mathcal{H}_+ \otimes \mathcal{V}_2(W^+)) \oplus (\mathcal{V}_2(W^+) \otimes \mathcal{V}_2(W^+))$$

(Since for any vector space V, $V^{\otimes 0} = \mathbb{R}$, we have omitted such factors in the above formula). Let $\mathbf{S} = \mathbf{S}_x^{Z(x)}$. Then, the linear map $\mathbf{S} : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(x)$ is given by

$$\mathbf{S}(\Psi) = \Psi, \qquad \mathbf{S}(E_1) = E_1, \qquad \mathbf{S}(E_2) = 0, \qquad \mathbf{S}(\Psi \otimes E_2) = E_1, \qquad \mathbf{S}(E_2 \otimes E_2) = E_1.$$

Example 4. We keep all notation from Examples 1-3. Suppose $u = \exp Y$, where $Y = tF$. We now compute the map $(u)_*$.

Note that by Lemma 4.1, we have $e_1(ux) = e_1(x)$. Also note that by Example 1, at x, the tangent vector to $U^+[x]$ coincides with $e_2(x)$. Recall that we are assuming that the foliation whose leaves are $U^+[x]$ is invariant under the geodesic flow. This implies that at the point ux, the tangent vector to the parabola $U^+[x]$ is $e_2(ux)$. Therefore,

$$e_1(ux) = e_1(x), \qquad e_2(ux) = te_1(x) + e_2(x).$$

Therefore,

$$P^+(x, ux)e_1(x) = e_1(ux), \qquad P^+(x, ux)e_2(x) = e_2(ux) = te_1(x) + e_2(x).$$

Suppose $\mathcal{U}$ is paramertized by (M', v'), where $M' = M'_{12}\Psi$, $v' = v'_1 e_1(x) + v'_2 e_2(x)$. Then

$$\exp[(I + M')Y] = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 + M'_{12}t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \qquad \exp(Y) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\exp[(I + M')Y](x + v') - \exp(Y)x = \begin{pmatrix} v'_1 + tv'_2 + tM'_{12} \\ v'_2 \\ 0 \end{pmatrix}$$

Let $\Psi' \in \mathrm{Hom}(\mathcal{V}_2(\mathrm{Lie}(U^+))(ux), \mathcal{V}_1(\mathrm{Lie}(\mathcal{G}_+))(ux))$ be the analogue of Ψ, but at the point ux. Then,

$$\begin{aligned} u_*(M', v') = u_*(M'_{12}\Psi, v'_1 e_1(x) + v'_2 e_2(x)) &= (M'_{12}\Psi', (v'_1 + tv'_2 + tM'_{12})e_1(x) + v'_2 e_2(x)) \\ &= (M'_{12}\Psi', (v'_1 + tM'_{12})e_1(ux) + v'_2 e_2(ux)) \end{aligned}$$

Then, in view of (6.11), $(S_{ux}^{Z(ux)} \circ u_*)(M', v') = (M_{12}\Psi', v_1 e_1(ux) + v_2 e_2(ux))$, where

$$v_1 = M'_{12}v'_2 + v'_1 + tM'_{12} + (v'_2)^2, \qquad v_2 = 0, \qquad M_{12} = M'_{12}.$$

Then, $(u)_* : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(ux)$ is given by

$$\begin{aligned} &(u)_*(\Psi) = \Psi' + tE_1, \quad (u)_*(E_1) = E_1, \quad (u)_*(E_2) = 0, \\ &\qquad\qquad (u)_*(\Psi \otimes E_2) = E_1, \quad (u)_*(E_2 \otimes E_2) = E_1. \end{aligned}$$

The dynamical system G_t. Suppose we fix some Lyapunov-admissible transversal $Z(x)$ for every $x \in X$. Suppose $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$ is adapted to $Z(x)$. Let

$$G_t(M, v) = S_{g_t x}^{Z(g_t x)}(g_t \circ M \circ g_t^{-1}, (g_t)_* v) \in \mathcal{H}_+(g_t x) \times W^+(g_t x). \tag{6.12}$$

Then, if $\mathcal{U}'$ is the affine subspace parametrized by (M, v) then $(M'', v'') = G_t(M, v) \in \mathcal{H}_+(g_t x) \times W^+(g_t x)$ parametrizes $g_t \mathcal{U}'$ and is adapted to $Z(g_t x)$. From the definition, we see that

$$G_{t+s} = G_t \circ G_s. \tag{6.13}$$

Also, it is easy to see that for $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$,

$$G_t(M, v) = (g_t \circ M' \circ g_t^{-1}, (g_t)_* v'), \qquad \text{where } (M', v') = S_x^{g_t^{-1} Z(g_t x)}(M, v). \tag{6.14}$$

The bundle $\mathbf{H}(x)$. Suppose we are given a Lyapunov adapted transversal $Z(x)$ at each $x \in X$. Let

$$\mathbf{H}(x) = \mathbf{S}_x^{Z(x)} \tilde{\mathbf{H}}(x)$$

denote the image of $\tilde{\mathbf{H}}(x)$ under $\mathbf{S}_x^{Z(x)}$. Then, if $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$ is adapted to $Z(x)$, then $\mathbf{j}(M, v) \in \mathbf{H}(x)$. We can also consider $(u)_*$ as defined in Lemma 6.2 (b) to be a map

$$(u)_* : \mathbf{H}(x) \to \mathbf{H}(ux).$$

Lemma 6.3.

(a) *Suppose $u'x = ux \in U^+[x]$ and $\mathbf{v} \in \mathbf{H}(x)$. Then $(u)_* \mathbf{v} = (u')_* \mathbf{v}$.*
(b) *We have $g_t \circ (u)_* \circ g_t^{-1} = (g_t u g_t^{-1})_*$.*

Proof. It is enough to prove (a) for $\mathbf{v} = \mathbf{j}(M, v)$ where $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$. Let $\mathcal{U}$ be the generalized subspace parametrized by (M, v). Then, $(u)_* \mathbf{v} = \mathbf{j}(M', v')$ where $(M', v') \in \mathcal{H}_+(ux) \times W^+(ux)$ is the (unique) parametrization of $\mathcal{U}$ adapted to $Z(ux)$. But then $(u')_* \mathbf{v}$ is also a parametrization of $\mathcal{U}$ adapted to $Z(ux)$. Therefore $(u')_* \mathbf{v} = (u)_* \mathbf{v}$.

The proof of (b) is essentially the same. $\square$

The bundle $\mathbf{H}(x)$ and the flow g_t. Let $Z(x)$ be an admissible transversal to $U^+(x)$ for every $x \in X$. Let $(g_t)_* : \mathbf{H}(x) \to \mathbf{H}(x)$ be given by

$$(g_t)_* = \mathbf{S}_{g_t x}^{Z(g_t x)} \circ \mathbf{f}_t \qquad \text{where } f_t(M, v) = (g_t \circ M \circ g_t^{-1}, (g_t)_* v), \tag{6.15}$$

$\mathbf{f}_t$ is the map induced by f_t on $\tilde{\mathbf{H}} \supset \mathbf{H}$, $(g_t)_*$ on the right-hand side is g_t acting on $W^+(x)$, g_t on the right-hand side is the natural map $\text{Lie}(U^+)(x) \to \text{Lie}(U^+)(g_t x)$, and $\mathbf{S}_x^Z$ is as in Lemma 6.2. Then $(g_t)_*$ is a linear map, and for $(M, v) \in \mathcal{H}_+(x) \times W^+(x)$,

$$(g_t)_*(\mathbf{j}(M, v)) = \mathbf{j}(G_t(M, v)). \tag{6.16}$$

Since $G_t \circ G_s = G_{t+s}$, and the linear span of $\mathbf{j}(\mathcal{H}_+(x) \times W^+(x))$ is $\tilde{\mathbf{H}}(x) \supset \mathbf{H}(x)$, it follows from (6.16) that $(g_t)_* \circ (g_s)_* = (g_{t+s})_*$.

Lemma 6.4. *Let $g_t : \mathbf{H}(x) \to \mathbf{H}(g_t x)$ and $\mathbf{f}_t : \tilde{\mathbf{H}}(x) \to \tilde{\mathbf{H}}(g_t x)$ be as in (6.15). Then the Lyapunov subspaces for g_t at x are the image under $\mathbf{S}_x^{Z(x)}$ of the Lyapunov subspaces of $\mathbf{f}_t$ at x, and the Lyapunov exponents of g_t are those Lyapunov exponents of $\mathbf{f}_t$ whose Lyapunov subspace at a generic point x is not contained in the kernel of $\mathbf{S}_x^{Z(x)}$.*

Proof. Let $\mathcal{V}_i(\tilde{\mathbf{H}})(x)$ and $\mathcal{V}_i(\mathbf{H})(x)$ denote the Lyapunov subspaces of the flow $\mathbf{f}_t$ and g_t respectively, and let λ_i denote the corresponding Lyapunov exponents. Then, for $\mathbf{v} \in \mathcal{V}_i(\tilde{\mathbf{H}})$, by the multiplicative ergodic theorem, for every $\epsilon > 0$,

$$\|g_t \mathbf{S}_x^{Z(x)} \mathbf{v}\|_Y = \|\mathbf{S}_{g_t x}^{Z(g_t x)} \mathbf{f}_t \mathbf{v}\|_Y \leq \|\mathbf{S}_{g_t x}^{Z(g_t x)}\|_Y \|\mathbf{f}_t \mathbf{v}\|_Y \leq C_\epsilon(x) C_1(g_t x) e^{\lambda_i t + \epsilon|t|}.$$

Taking $t \to \infty$ and $t \to -\infty$ we see that $\mathbf{S}_x^{Z(x)} \mathbf{v} \in \mathcal{V}_i(\mathbf{H})(x)$. □

6.1. Approximation of generalized subspaces and the map $\mathcal{A}(\cdot,\cdot,\cdot,\cdot)$.

Hausdorff distance between generalized subspaces. For $x \in X$, and two generalized subspaces $\mathcal{U}'$ and $\mathcal{U}''$, let $hd_x(\mathcal{U}', \mathcal{U}'')$ denote the Hausdorff distance using the metric derived from $\|\cdot\|_Y$ between $\mathcal{U}' \cap B(x,1)$ and $\mathcal{U}'' \cap B(x,1)$.

Lemma 6.5. *Suppose $hd_x(U^+[x], \mathcal{U}(M,v)) \leq 1$.*

(a) *We have for some absolute constant $C > 0$,*

$$hd_x(U^+[x], \mathcal{U}(M,v)) \leq C \max(\|v\|_Y, \|M\|_Y).$$

Also if $(M,v)_x$ is adapted to $Z(x)$, then there exists $c(x) > 0$ such that

$$hd_x(U^+[x], \mathcal{U}(M,v)) \geq c(x) \max(\|v\|_Y, \|M\|_Y).$$

(b) *For some $c_1(x) > 0$ a.e., we have*

$$c_1(x)\|\mathbf{j}(M,v)\|_Y \leq hd_x(U^+[x], \mathcal{U}(M,v)) \leq c_1(x)^{-1}\|\mathbf{j}(M,v)\|_Y.$$

Proof. Part (a) is immediate from the definitions. To see (b) note that part (a) implies that $\max(\|M\|_Y, \|v\|_Y) = O(1)$, and thus all the higher order terms in $\mathbf{j}(M,v)$ which are polynomials in M_{ij} and v_j, are have size bounded by a constant multiple of the the size of the first order terms, i.e. by $\max(\|M\|_Y, \|v\|_Y)$. □

We will also use the following crude estimate:

Lemma 6.6. *There exists a function $C(x)$ finite almost everywhere such that the following holds: Suppose $x \in X$, $\mathcal{U} \subset W^+(x)$ and $\mathcal{U}' \subset W^+(x)$ are two affine subspaces, and $t > 0$. Then,*

$$hd_{g_t x}((g_t)_* \mathcal{U}, (g_t)_* \mathcal{U}') \leq C(x) e^{2t} hd_x(\mathcal{U}, \mathcal{U}'),$$

provided the quantity on the left is at most 1. Here, $C(x) > 0$ is finite a.e.

Proof. This follows from the fact that the action of g_t on W^+ can expand by at most e^{2t}, see also Lemma 3.5. $\square$

Motivation. Suppose $u \in \mathcal{B}$ and $t > 0$ is not too large. Then, for some $u' \in \mathcal{B}$ the generalized subspaces $U^+[g_t u q_1]$ and $U^+[g_t u' q_1']$ will be nearby. These subspaces are not on the same leaf of W^+ (even though the leaf $W^+[g_t u' q_1']$ containing $U^+[g_t u' q_1']$ gets closer to the leaf $W^+[g_t u q_1]$ containing $U^+[g_t u q_1]$ as $t \to \infty$). It is convenient to find a way to "project" $U^+[g_t u' q_1']$ to $W^+[g_t u q_1]$. In particular, we want the projection to be again a generalized subspace (i.e. an orbit of a subgroup of $\mathcal{G}_+(g_t u q_1)$). We also want the projection to be exponentially close to the original generalized subspace $U^+[g_t u q_1']$. Furthermore, in order to carry out the program outlined in the beginning of §5, we want the pair (M'', v'') parametrizing the projection to be such that $\mathbf{j}(M'', v'') \in \mathbf{H}(q_t u q_1)$ depends polynomially on $P^-(q_1, q_1')$. Then it will depend linearly on on $F(q) - F(q')$ since any fixed degree polynomial in $P^-(q_1, q_1')$ can be expressed as a linear function of $F(q) - F(q')$ as long as r in the definition of $\mathcal{L}_{ext}(q)^{(r)}$ is chosen large enough.

More precisely, we need the following:

Lemma 6.7. *We can choose r and s sufficiently large (depending only on the Lyapunov spectrum) so that there exists a linear map $\mathcal{A}(q_1, u, \ell, t) : \mathcal{L}_{ext}(q)^{(r)} \to \mathbf{H}(g_t u q_1)$ satisfying the equivariance property*

$$\mathcal{A}(q_1, u, \ell + \ell', t + t') = g_{t'} \circ \mathcal{A}(q_1, u, \ell, t) \circ g_{\ell'}. \tag{6.17}$$

and functions $C_1 : X \to \mathbb{R}$ and $C_2 : X \to \mathbb{R}$ finite almost everywhere such that the following holds: Suppose $\delta > 0$, and ℓ is sufficiently large depending on δ. Let $q = g_{-\ell} q_1$ (see Figure 1). For all $u \in \mathcal{B}$, any $q' \in W^-[q]$ satisfying (5.6) and (5.7), and any $t > 0$ such that there exists $u' \in 2\mathcal{B}$ with

$$hd_{g_t u q_1}(U^+[g_t u q_1], U^+[g_t u' q_1']) \le 1, \tag{6.18}$$

where $q_1' = g_\ell q'$, we have

$$hd_{g_t u q_1}(U^+[g_t u' q_1'], \mathcal{U}(M'', v'')) \le C_1(q_1) C_2(u q_1) e^{-\alpha_1 \ell} \tag{6.19}$$

where $\alpha_1 > 0$ depends only on s and the Lyapunov spectrum, $\mathcal{U}(M'', v'') \subset W^+(g_t u q_1)$ is the affine subspace parametrized by (M'', v''), and (M'', v'') are such that

$$\mathcal{A}(q_1, u, \ell, t)(F(q') - F(q)) = \mathbf{j}(M'', v''), \tag{6.20}$$

Hence,

$$\begin{aligned} C(g_t u q_1)^{-1} \|\mathcal{A}(q_1, u, \ell, t)(F(q') - F(q))\|_Y \le \\ \le hd_{g_t u q_1}(U^+[g_t u q_1], U^+[g_t u' q_1']) \le C(g_t u q_1) \|\mathcal{A}(q_1, u, \ell, t)(F(q') - F(q))\|_Y, \end{aligned} \tag{6.21}$$

where $C : X \to \mathbb{R}^+$ is finite a.e.

Lemma 6.7 is proved by constructing a linear map $\tilde{P}_s : W^+(u'q_1) \to W^+(uq_1)$ with nice properties; then the approximating subspace $\mathcal{U}(M'', v'')$ is given by $g_t \tilde{P}_s U^+[u'q_1']$. The construction is technical, and is postponed to §6.2*. Then, Lemma 6.7 is proved in §6.3*. From the proof, we will also deduce the following lemma (which will be used in §12):

Lemma 6.8. *For every $\delta > 0$ there exists a compact set K with $\nu(K) > 1 - \delta$ so that the following holds: Suppose $q \in K$, $\ell > 0$ is sufficiently large depending on δ, and suppose $q' \in W^-[q] \cap K$ is such that (5.6) and (5.7) hold. Let $q_1 = g_\ell q$, $q_1' = g_\ell q'$ (see Figure 1). Fix $u \in U^+[q_1]$, and suppose $t > 0$ is such that there exists $u' \in 2\mathcal{B}$ so that*

$$hd_{g_t u q_1}(U^+[g_t u q_1], U^+[g_t u' q_1']) \le \epsilon \ll 1.$$

Furthermore, suppose q_1, q_1', $g_t u q_1$ and $g_t u' q_1'$ all belong to K. Let

$$A_t = U^+[g_t u q_1] \cap B(g_t u q_1, 1)$$

$$A_t' = U^+[g_t u' q_1'] \cap B(g_t u q_1, 1)$$

Then,

$$\kappa^{-1} |g_{-t} A_t| \le |g_{-t} A_t'| \le \kappa |g_{-t} A_t|,$$

where κ depends only on the Lyapunov spectrum. Also,

$$hd(g_{-t} A_t, g_{-t} A_t') \le e^{-\alpha \ell},$$

where $hd(\cdot, \cdot)$ denotes the Hausdorff distance, and α depends only on the Lyapunov spectrum.

This lemma will also be proved in §6.3*. Finally, we state and prove Proposition 6.9 which tells us when the inductive procedure outlined in §2.3 stops.

Recall the notational conventions (2.2).

Proposition 6.9. *For any $\ell > 0$ we have $\|\mathcal{A}(q_1, u, \ell, t)\|_Y \to \infty$ as $t \to \infty$ for almost all $q_1 \in X$ and $u \in U$, unless $\mathcal{L}(q_1) \subset U(q_1)$ for almost all q_1. (Here we identify the linear subspace $\mathcal{L}(q_1)$ with the group of translations by vectors in this subspace).*

The proof of Proposition 6.9 will rely on the following

Lemma 6.10. *Let h_t denote the automprphism of the affine group $\mathcal{G}_+(x)$ which is the identity on the linear part and multiplication by e^{2t} on the translational part. Let $U^+(x)$ be any subgroup of $\mathcal{G}_+(x)$. Then,*

$$\bigcap_{t \in \mathbb{R}} (h_t U^+ h_t^{-1})[x] = U_2^+[x], \tag{6.22}$$

where $U_2^+(x)$ is the intersection of $U^+(x)$ with the pure translations in $\mathcal{G}_+(x)$.

Proof of Lemma 6.10. Let L denote the left-hand-side of (6.22). Let

$$\mathfrak{u} = \{Y \in \mathrm{Lie}(U^+)(x) \quad : \quad (\exp Y)x \in L\}.$$

Then, for all t, $(h_t)_* \mathfrak{u} = \mathfrak{u}$, where $(h_t)_*$ denotes the adjoint action of h_t on $\mathrm{Lie}(\mathcal{G}_+)(x)$, which is in this case the identity on the linear part, and multiplication by e^{2t} on the pure translation part. Suppose $Y \in \mathfrak{u}$. We can write $Y = Y_1 + Y_2$, where $Y_1 \in \mathrm{Lie}(Q_+)(x)$ is the linear part and Y_2 is a pure translation. Then, for all t, $Y_1 + e^{2t} Y_2 \in \mathfrak{u} \subset \mathrm{Lie}(U^+)(x)$. Therefore, $Y_2 \in \mathrm{Lie}(U_2^+)(x)$. Thus,

$$\mathfrak{u} \subset \mathrm{Lie}(U_1^+)(x) + \mathrm{Lie}(U_2^+)(x),$$

where $U_1^+(x) = U^+(x) \cap Q_+(x)$ denotes the linear part of $U^+(x)$. Note that $U_2^+(x)$ is a normal subgroup of $U^+(x)$, and thus, $\exp \mathfrak{u} \subset U_2^+(x) U_1^+(x)$. Since $U_1^+(x)$ fixes the origin x, we have

$$L = (\exp \mathfrak{u})[x] \subset U_2^+ U_1^+[x] = U_2^+[x].$$

Conversely, it is clear that $U_2^+[x] \subset L$ (since $U_2^+[x]$ is a linear subspace and is thus invariant under dilations). $\square$

Proof of Proposition 6.9. Let $L^-[q] \subset W^-[q]$ denote the smallest real-algebraic subset containing, for some $\epsilon > 0$, the intersection of the ball of radius 1 with the support of the measure $\nu_{W^-(q)}$, which is the conditional measure of ν along $W^-[q]$. Then, $L^-[q]$ is g_t-equivariant. Since the action of g_{-t} is expanding along $W^-[q]$, we see that for almost all q and any $\epsilon > 0$, $L^-[q]$ is the smallest real-algebraic subset of $W^-(q)$ such that $L^-[q]$ contains $E_\epsilon^-[q]$, where $E_\epsilon^-[q] \equiv \mathrm{support}(\nu_{W^-(q)}) \cap B(q,\epsilon)$. Let $E_\epsilon^-(q) = E_\epsilon^-[q] - q$, $L^-(q) = L^-[q] - q$.

Fix $q_1 \in X$ and suppose $\|\mathcal{A}(q_1, u, \ell, t)\|_Y$ is bounded as a function of t for almost all $u \in U$. By Lemma 6.7, this implies that for some $\epsilon > 0$ and for every $q_1' \in E_\epsilon^-[q_1]$,

$$\pi_{W^+(q_1)}(U^+[q_1']) \subset U^+[q_1], \tag{6.23}$$

where for x near q_1, $\pi_{W^+(q_1)}(x)$ is the unique point in $W^+[q_1] \cap AW^-[x]$.

Let $\hat{\pi}^+ : W(x) \to W^+(x)$ and $\hat{\pi}^- : W(x) \to W^-(x)$ denote the maps

$$\hat{\pi}_{q_1}^+(v) = (1,0) \otimes v, \qquad \hat{\pi}_{q_1}^-(v) = (0,1) \otimes \pi_{q_1}^-(v),$$

where $\pi_{q_1}^-$ is as in (2.2). Let $n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \subset N \subset SL(2,\mathbb{R})$.

Claim 6.11. *Suppose $q_1' \in W^-[q_1]$. Then, we have*

$$\pi_{W^+(q_1)}(n_t q_1') = q_1 + (1,0) \otimes t(1+ct)^{-1} (\hat{\pi}_{q_1}^-)^{-1}(q_1' - q_1),$$

where $c = p(v) \wedge p(Im\, q_1)$.

Proof of Claim. Since $q_1' \in W^-[q_1]$, we can write $q_1' = q_1 + (0,1) \otimes v$, where $v \wedge \mathrm{Re}\, q_1 = 0$. Then,

$$n_t q_1' = (1,0) \otimes (\mathrm{Re}\, q_1 + t(\mathrm{Im}\, q_1 + v)) + (0,1) \otimes (\mathrm{Im}\, q_1 + v).$$

Let

$$w = v + ct(1+ct)^{-1}\mathrm{Im}\, q_1.$$

Then, $p(w) \wedge p(\mathrm{Re}\,(n_t q_1)) = 0$, and thus, $(0,1) \otimes w \in W^-(n_t q_1)$. Therefore,

$$n_t q_1 - w = (1,0) \otimes (\mathrm{Re}\, q_1 + t(\mathrm{Im}\, q_1 + v)) + (0,1) \otimes (1+ct)^{-1}\mathrm{Im}\, q_1 \in W^-[n_t q_1].$$

We have $\begin{pmatrix} (1+ct)^{-1} & 0 \\ 0 & 1+ct \end{pmatrix} \in A$. Therefore,

$$(1,0) \otimes (1+ct)^{-1}(\mathrm{Re}\, q_1 + t(\mathrm{Im}\, q_1 + v)) + (0,1) \otimes \mathrm{Im}\, q_1 \in AW^-[n_t q_1]. \tag{6.24}$$

It is easy to check that (6.24) is in $W^+[q_1]$. Therefore,

$$\begin{aligned} \pi_{W^+(q_1)}(n_t q_1') = (1,0) \otimes (1+ct)^{-1}(\mathrm{Re}\, q_1 + t(\mathrm{Im}\, q_1 + v)) + (0,1) \otimes \mathrm{Im}\, q_1 = \\ = q_1 + (1,0) \otimes t(1+ct)^{-1}(\mathrm{Im}\, q_1 + v'), \end{aligned}$$

where $v' \in H^1_\perp$ is such that $v = c\,\mathrm{Re}\, q_1 + v'$. Also,

$$\mathrm{Im}\, q_1 + v' = (\pi^-_{q_1})^{-1}(v) = (\hat{\pi}^-_{q_1})^{-1}(q_1' - q_1).$$

This completes the proof of the claim. □

Proof of Proposition 6.9 continued. Suppose $q_1' \in E^-_\epsilon[q_1]$. Without loss of generality, we may assume that $\epsilon > 0$ is small enough so that the constant c in Claim 6.11 satisfies $c < 1/2$. Now choose t so that $t(1+ct)^{-1} = 1$. Since $n_t q_1' \in U^+[q_1']$, (6.23) and Claim 6.11 imply that for all $q_1' \in E^-_\epsilon[q_1]$,

$$q_1 + (1,0) \otimes (\hat{\pi}^-_{q_1})^{-1}(q_1' - q_1) \in U^+[q_1].$$

Hence,

$$L^+[q_1] \equiv q_1 + (1,0) \otimes (\hat{\pi}^-_{q_1})^{-1}(L^-(q_1)) \subset U^+[q_1].$$

Now suppose that for of q_1 in a set Y of positive measure, $\|\mathcal{A}(q_1, u, \ell, t)\|_Y$ is bounded as a function of t. Let Ω be the set such that for $q_1 \in \Omega$, $g_t q_1$ spends a positive proportion of the time in Y. Then, by the ergodicity of g_t, Ω is conull. For $q_1 \in \Omega$, we have, for a positive fraction of t,

$$L^+[g_t q_1] \subset U^+[g_t q_1]. \tag{6.25}$$

Let $A(x,t)$ denote the Kontsevich-Zorich cocycle. Then g_t acts on W^+ by $e^t A(x,t)$ and on W^- by $e^{-t}A(x,t)$. Therefore, $L^-(g_t q_1) = e^{-t}A(x,t)L^-(q_1)$, and thus $L^+[g_t q_1] = e^{-t}A(x,t)L^+[g_t q_1]$. Also, we have $U^+[g_t q_1] = e^t A(x,t) U^+[q_1]$. Thus, for a positive measure set of t, we have

$$L^+[q_1] \subset e^{2t}U^+[q_1] = (h_t U^+ h_t^{-1})[q_1], \tag{6.26}$$

where h_t is as in Lemma 6.10. (In the derivation of (6.26) we were careless with the $SL(2,\mathbb{R})$ orbit direction, but this does not matter since U^+ contains N). Since both

sides of (6.26) depend analytically on t, we see that (6.26) holds for all t. Then, by Lemma 6.10, $L^+[q_1] \subset U_2^+[q_1]$. Thus,

$$q_1 + (1,0) \otimes \hat{\pi}_{q_1}^{-1}(E_\epsilon^-(q_1)) \subset L^+[q_1] \subset U_2^+[q_1]$$

Since $U_2^+[q_1]$ is an affine subspace and $\mathcal{L}^-(q_1)$ is the smallest subspace containing $E_\epsilon^-(q_1)$ it follows that

$$q_1 + (1,0) \otimes \hat{\pi}_{q_1}^{-1}(\mathcal{L}^-(q_1)) \subset L^+[q_1] \subset U_2^+[q_1].$$

Thus, for almost all q_1, $\mathcal{L}(q_1) \subset U(q_1)$. □

6.2*. **Construction of the map $\mathcal{A}(q_1, u, \ell, t)$.**

Motivation. To construct the generalized subspace $\mathcal{U} = \mathcal{U}(M'', v'')$ of Lemma 6.7 we will construct a linear map $\tilde{P}_s(uq_1, q_1') : W^+(uq_1) \to W^+(\pi_{W^+(q_1')}(uq_1))$, and let $\mathcal{U} = \tilde{P}_s(uq_1, q_1')^{-1} U^+[q_1']$. We want $\tilde{P}_s(uq_1, q_1')$ to have the following properties:

- (P1) $\tilde{P}_s(uq_1, q_1')$ depends only on $W^+[q_1']$, i.e. for $z \in W^+[q_1']$, $\tilde{P}_s(uq_1, z) = \tilde{P}_s(uq_1, q_1')$.
- (P2) $\|\tilde{P}_s(g_t uq_1, g_t u' q_1')^{-1} - I\|_Y = O(e^{-\ell})$ for all $t > 0$ such that (6.18) holds.
- (P3) The (entries of the matrix) $\tilde{P}_s(g_t uq_1, q_1')$ are polynomials of degree at most s in (the entries of the matrix) $P^-(q_1, q_1')$.
- (P4) The generalized subspace $\mathcal{U} = \tilde{P}_s(uq_1, q_1')^{-1} U^+[q_1']$ can be parametrized by $(M'', v'') \in \mathcal{H}_+(uq_1) \times W^+(uq_1)$ (and not by an element of $\mathcal{H}(uq_1) \times W^+(uq_1)$.

The construction will take place in several steps.

The map $\hat{P}(x, y)$. There exists a set K of full measure such that each point x in K is Lyapunov-regular with respect to the bundle W^+, i.e.

$$W^+(x) = \bigoplus_i \mathcal{V}_i(x),$$

where $\mathcal{V}_i(x) = \mathcal{V}_i(W^+)(x)$ are the Lyapunov subspaces, and the multiplicative ergodic theorem holds. We have the flag

$$\{0\} \subset V_1(x) \subset \cdots \subset V_n(x) = W^+(x), \tag{6.27}$$

where $V_j(x) = \bigoplus_{i=1}^j \mathcal{V}_i(x)$. Note that if $y \in W^+[x]$ is also Lyapunov-regular, then the flag (6.27) at y agrees with the flag at x. Thus, we may define (6.27) at any point x such that $W^+[x]$ contains a regular point.

We have, for each i,

$$W^+(x) = V_i(y) \oplus \bigoplus_{j=i+1}^n \mathcal{V}_j(x). \tag{6.28}$$

Let $\hat{P}_i : \mathcal{V}_i(x) \to W^+(\pi_{W^+(y)}(x))$ be the map taking $v \in \mathcal{V}_i(x)$ to its $V_i(y)$ component under the decomposition (6.28). Let $\hat{P}(x, y) : W^+(x) \to W^+(\pi_{W^+(y)}(x))$ be the linear map which agrees with $\hat{P}_i$ on each $\mathcal{V}_i(x)$. (By our conventions, $\hat{P}(x, y)$ takes the origin

x of $W^+(x)$ to $\pi_{W^+(y)}(x)$, where, as in the proof of Proposition 6.9, $\pi_{W^+(y)}(x)$ is the unique point in $W^+[y] \cap AW^-[x]$.)

We have

$$\hat{P}(g_t x, g_t y) = g_t \hat{P}(x, y), \tag{6.29}$$

and

$$\hat{P}(x, y) V_i(x) = V_i(y). \tag{6.30}$$

The following lemma essentially states that the map $\hat{P}(uq_1, q_1')$ has properties (P1) and (P2).

Lemma 6.12. *Suppose q, q', ℓ, t, u and u' are as in Lemma 6.7. (Note that $U^+[g_t u' q_1'] = g_t U^+[u' q_1']$ is locally independent of $u' \in U^+(q_1')$.).*

Let $\widehat{\mathcal{U}} = \hat{P}(uq_1, q_1')^{-1}(U^+[q_1'])$. Then $\widehat{\mathcal{U}} \subset W^+(q_1)$ is an generalized affine subspace, and

$$hd_{g_t uq_1}(g_t \widehat{\mathcal{U}}, U^+[g_t u' q_1']) \le C(q_1) C(uq_1) e^{-\alpha(t+\ell)},$$

where $\alpha > 0$ depends only on the Lyapunov spectrum, and $C : X \to \mathbb{R}^+$ is finite almost everywhere.

Proof. In this proof, we write $\mathcal{V}_i(x)$ for $\mathcal{V}_i(W^+)(x)$ and $V_i(x)$ for $V_i(W^+)(x)$. Let $\alpha_0 = \min |\lambda_i - \lambda_j|$ (where the λ_i are the Lyapunov exponents for W^+). Choose $0 < \epsilon < \alpha_0/100$. We first assume that $g_t x \in K$ where $\nu(K) > 1 - \epsilon/2$ and also

$$d(\mathcal{V}_i(x), \mathcal{V}_j(x)) > \sigma \tag{6.31}$$

for all $x \in K$ and all $i \neq j$, and $\sigma > 0$ depends only on ν and ϵ. We claim that in view of (5.7), for sufficiently large ℓ (depending on δ),

$$d(V_i(q_1), V_i(q_1')) \le C_1(q) e^{-(\alpha_0 - \epsilon)\ell}. \tag{6.32}$$

Indeed, by Lemma 4.1, we can make a basis for $V_i(q_1')$ consisting of vectors of the form $v_i + \sum_{j>i} v_j'$, where $v_i \in \mathcal{V}_i(q_1)$, $v_j' \in \mathcal{V}_j(q_1)$. By (5.7), $\|v_j'\|_Y \le C(\delta)\|v_i\|_Y$. Then, by the multiplicative ergodic theorem, we may write

$$\frac{g_t(v_i + \sum_{j>i} v_j')}{\|g_t(v_i + \sum_{j>i} v_j')\|_Y} = \frac{g_t v_i}{\|g_t v_i\|_Y} + w,$$

where

$$\|w\|_Y \le C_1(q) C(\delta) e^{-(\lambda_i - \lambda_j + \epsilon)\ell} \le C_1(q) C(\delta) e^{-(\alpha_0 - \epsilon)\ell}.$$

This proves (6.32). Since $V_i(uq_1) = V_i(q_1)$ and $V_i(u'q_1') = V_i(q_1')$, we have, in view of (6.32) and after applying the same argument,

$$d(V_i(g_t uq_1), V_i(g_t u' q_1') \le C_1(q) C_2(uq_1) e^{-(\alpha_0 - \epsilon)\ell} e^{-(\alpha_0 - \epsilon)t}. \tag{6.33}$$

Then, by (6.31), (6.33), and the definition of $\hat{P}(x, y)$,

$$\|\hat{P}(g_t uq_1, g_t u' q_1')^{-1} - I\|_Y \le C_1(q_1) C_2(uq_1) C(\sigma) e^{-\alpha'(\ell+t)}, \tag{6.34}$$

where α' depends only on the Lyapunov spectrum.

Note that $\widehat{\mathcal{U}}$ is the orbit of a subgroup $\hat{U}$ of $\mathcal{G}(uq_1)$ whose Lie algebra is

$$\hat{P}(uq_1, q_1')_*^{-1} \operatorname{Lie}(U^+)(q_1')$$

(and we are using the notation (6.9)). By (6.30) and the fact that $\operatorname{Lie}(U^+)(q_1') \in \mathcal{G}_+(q_1')$ we have $\operatorname{Lie}(\hat{U}) \in \mathcal{G}_+(uq_1)$. Thus, $\widehat{\mathcal{U}}$ is a generalized subspace.

Since $U^+[q_1']$ is a generalized subspace, for all $u' \in U^+(q_1)$, $U^+[q_1] = U^+[u'q_1]$. We have

$$g_t\widehat{\mathcal{U}} = g_t\hat{P}(uq_1, u'q_1')^{-1}U^+[u'q_1'] = \hat{P}(g_tuq_1, g_tu'q_1')^{-1}U^+[g_tu'q_1'].$$

Therefore, by (6.34),

$$hd_{g_tuq_1}(g_t\widehat{\mathcal{U}}, U^+[g_tu'q_1']) \le C_1(q_1)C_2(uq_1)e^{-\alpha'(\ell+t)}, \tag{6.35}$$

This proves the lemma, assuming $g_tuq_1 \in K$. To prove the lemma in general, note that for almost all $x \in X$, by the Birkhoff ergodic theorem, for all t sufficiently large (depending on x),

$$|\{t' \in [0,t] \quad : \quad g_{t'}x \in K\}| \ge (1-\epsilon)t.$$

Therefore, we for any sufficiently large t can find t' such that $g_{t'}uq_1 \in K$ and $t \ge t' > (1-\epsilon)t$. Now the lemma follows with $\alpha = \alpha' - 2\epsilon$ from applying (6.35) to t' instead of t and using Lemma 3.5. $\square$

Motivation. Suppose $q_1 \in \tilde{X}$, $u \in U^+(q_1)$, $q_1' \in W^-[q_1]$. In view of Lemma 6.12, $\hat{P}(uq_1, q_1')$ has properties (P1) and (P2). We claim that it does not in general have the properties (P3) and (P4).

Let

$$\hat{Q}(uq_1; q_1') = \hat{P}(uq_1, q_1')^{-1}P^-(q_1, q_1') \circ P^+(uq_1, q_1), \tag{6.36}$$

so that

$$\hat{P}(uq_1, q_1')\hat{Q}(uq_1; q_1') = P^-(q_1, q_1')P^+(uq_1, q_1). \tag{6.37}$$

Then, $\hat{Q}(uq_1; q_1') : W^+(uq_1) \to W^+(uq_1)$ and $\hat{Q}(uq_1; q_1')V_i(uq_1) = V_i(uq_1)$, hence $\hat{Q}(uq_1; q_1') \in Q(uq_1)$.

We now show how to compute $\hat{P}(uq_1, q_1')$ and $\hat{Q}(uq_1; q_1')$ in terms of $P^+ = P^+(uq_1, q_1)$ and $P^- = P^-(q_1, q_1')$. In view of Lemma 4.2, P^+ is upper triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(uq_1)$. Also by Lemma 4.2 applied to P^- instead of P^+, P^- is upper triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(q_1)$. Therefore, since P^+ takes $\mathcal{V}_i(uq_1)$ to $\mathcal{V}_i(q_1)$, $(P^+)^{-1}P^-P^+$ is upper triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(uq_1)$.

Let $\hat{P} = \hat{P}(uq_1, q_1')$, $\hat{Q} = \hat{Q}(uq_1; q_1')$. Then, in view of the definition of $\hat{P}$, $\hat{P}$ is lower triangular with 1's along the diagonal in terms of a basis adapted to $\mathcal{V}_i(uq_1)$ (and we identify $W^+(q_1')$ with $W^+(uq_1)$ using the Gauss-Manin connection). Also,

since $\hat{Q}$ preserves the flag $V_i(uq_1)$, $\hat{Q}$ is upper triangular in terms of the basis adapted to $\mathcal{V}_i(uq_1)$. Thus, (6.37) can we written as

$$\hat{P}\hat{Q} = P^- P^+ = P^+((P^+)^{-1}P^-P^+) \tag{6.38}$$

Recall that the Gaussian elimination algorithm shows that any matrix A in neighborhood of the identity I can be written uniquely as $A = LU$ where L is lower triangular with 1's along the diagonal and U is upper triangular. Thus, $\hat{P} = \hat{P}(uq_1, q_1')$ and $\hat{Q} = \hat{Q}(uq_1; q_1')$ are the L and U parts of the LU decomposition of the matrix $A = P^-(q_1, q_1')P^+(uq_1, q_1)$. (Note that we are given $A = U'L'$ where $U' = P^+$ is upper triangular and $L' = (P^+)^{-1}P^-P^+$ is lower triangular, so we are really solving the equation $LU = U'L'$ for L and U).

Since the Gaussian elimination algorithm involves division, the entries of $\hat{P}(uq_1, q_1')$ is are rational functions of the entries of $P^+(uq_1, q_1)$ and $P^-(q_1, q_1')$, but not in general polynomials. This means that $\hat{P}(uq_1, q_1')$ does not in general have property (P3). Also, the diagonal entries of $\hat{Q}(uq_1; q_1')$ are not 1. This eventually translates to the failure of the property (P4). Both problems are addressed below.

The maps $\hat{P}_s(uq_1, q_1')$ and $\tilde{P}_s(uq_1, q_1')$. For $s > 1$, let $\hat{Q}_s(uq_1; q_1')$ be the order s Taylor approximation to $\hat{Q}(uq_1; q_1')$, where the variables are the entries of $P^-(q_1, q_1')$ (and u, q_1 and the entries of $P^+(uq_1, q_1)$ are considered constants). Then, $\hat{Q}_s = \hat{Q}_s(uq_1; q_1') \in Q(uq_1)$. We may write

$$\hat{Q}_s = D_s + \tilde{Q}_s \tag{6.39}$$

where D_s preserves all the subspaces $\mathcal{V}_i(uq_1)$ and $\tilde{Q}_s = \tilde{Q}_s(uq_1; q_1') \in Q_+(uq_1)$. Let $\tilde{P}_s(uq_1, q_1')$ be defined by the relation:

$$\tilde{P}_s(uq_1, q_1')^{-1} = \tilde{Q}_s(uq_1; q_1')P^+(uq_1, q_1)^{-1}P^-(q_1, q_1')^{-1}. \tag{6.40}$$

Motivation. We will effectively show that for s sufficiently large, the map $\tilde{P}_s(uq_1, q_1')$ has the properties (P1),(P2), (P3) and (P4).

We have

$$\tilde{P}_s(uq_1, q_1')^{-1}V_i(q_1') = V_i(uq_1).$$

As a consequence,

$$\tilde{P}_s(uq_1, q_1')^{-1} \circ Y \circ \tilde{P}_s(uq_1, q_1') \in \mathcal{G}_+(uq_1) \qquad \text{for all } Y \in \mathcal{G}_+(q_1').$$

Thus, for any subalgebra L of $\mathrm{Lie}(\mathcal{G}_+)(q_1')$, $\tilde{P}_s(uq_1, q_1')_*(L)$ is a subalgebra of $\mathrm{Lie}(\mathcal{G}_+)(uq_1)$, where $\tilde{P}_s(uq_1, q_1')_*^{-1} : \mathrm{Lie}(\mathcal{G}_+)(q_1') \to \mathrm{Lie}(\mathcal{G}_+)(uq_1)$ is as in (6.9).

The map $i_{u,q_1,s}$.

Motivation. We want $i_{u,q_1,s} : \mathcal{L}_{ext}(q_1) \to \mathcal{H}_+(uq_1) \times W^+(uq_1)$ to be such that

$$i_{u,q_1,s}(\mathfrak{P}(q_1') - \mathfrak{P}(q_1)) = (M_s, v_s),$$

where $(M_s, v_s) \in \mathcal{H}_+(uq_1) \times W^+(uq_1)$ parametrizes the approximation $\tilde{P}_s(uq_1, q_1')^{-1}U^+[q_1']$ to $U^+[q_1']$ constructed above. Furthermore, we want $i_{u,q_1,s}$ to be a polynomial map of degree at most s in the entries of $\mathfrak{P}(q_1') - \mathfrak{P}(q_1)$.

By Proposition 4.4 (a), we have

$$\operatorname{Lie}(U^+)(q_1') = P^-(q_1, q_1')_*(\operatorname{Lie}(U^+)(q_1)),$$

where we used the notation (6.9). Let $\mathcal{U}_s' = \tilde{P}_s(uq_1, q_1')^{-1}U^+[q_1']$. We first find $(M_s', v_s') \in \mathcal{H}(q_1) \times W^+(q_1)$ which parametrizes $\mathcal{U}_s'$. Let

$$v_s' = \tilde{P}_s(uq_1, q_1')^{-1}q_1' \in \mathcal{U}_s' \subset W^+[q_1].$$

Then $\mathcal{U}_s' = U_s'[v_s']$ where the subgroup U_s' of $\mathcal{G}_+(q_1)$ is such that

$$\operatorname{Lie}(U_s') = \tilde{P}_s(uq_1, q_1')_*^{-1} \circ P^-(q_1, q_1')_*(\operatorname{Lie}(U^+)(q_1)).$$

Let

$$M_s' = \tilde{P}_s(uq_1, q_1')_*^{-1} \circ P^-(q_1, q_1')_* - I \in \operatorname{Hom}(\operatorname{Lie}(U^+)(q_1), \operatorname{Lie}(\mathcal{G}_+)(q_1)) = \mathcal{H}(q_1).$$

Then (M_s', v_s') parametrizes $\mathcal{U}_s'$. By (6.40),

$$M_s' = \tilde{Q}_s(uq_1; q_1')_* \circ P^+(uq_1, q_1)_*^{-1} - I.$$

Since both $\tilde{Q}_s(uq_1; q_1')$ and $P^+(uq_1, q_1)^{-1}$ are unipotent upper triangular, $M_s' \in \mathcal{H}_+(q_1)$. Now let

$$(M_s, v_s) = u_*(M_s', v_s'),$$

where u_* is defined as in (6.10). Then, $(M_s, v_s) \in \mathcal{H}_+(uq_1) \times W^+(uq_1)$ parametrizes $\mathcal{U}_s'$ as desired. Note that in view of (6.10),

$$M_s = M_s' \circ P^{-1}(ux, x)_* = \tilde{Q}_s(uq_1; q_1')_* - I.$$

where $\tilde{Q}_s(\cdot; \cdot)$ is as in (6.39), and we are using the notation (6.9). Since $\tilde{Q}_s(uq_1; q_1') \in Q_+(uq_1)$, $M_s \in \mathcal{H}_+(uq_1)$.

Note that by (5.4), we can recover q_1 from $\mathfrak{P}(q_1)$. Also, since by Proposition 4.4 (a), for $q_1' \in W^-[q_1]$,

$$\operatorname{Lie}(U^+)(q_1') = P^-(q_1, q_1')_* \operatorname{Lie}(U^+)(q_1) = (\mathfrak{P}(q_1') \circ \mathfrak{P}(q_1)^{-1})_* \operatorname{Lie}(U^+)(q_1), \tag{6.41}$$

we can reconstruct $U^+(q_1')$ if we know $\mathfrak{P}(q_1)$, $U^+(q_1)$ and $\mathfrak{P}(q_1')$. Now let $i_{u,q_1,s} : \mathcal{L}_{ext}(q_1) \to \mathcal{H}_+(uq_1) \times W^+(uq_1)$ be the map taking $\mathfrak{P}(q_1') - \mathfrak{P}(q_1)$ to (M_s, v_s). This is a polynomial map, since both q_1' and $\operatorname{Lie}(U^+)(q_1')$ can be recovered from $\mathfrak{P}(q_1')$ using (5.4) and (6.41). (Note that q_1 is considered fixed here, so knowing $\mathfrak{P}(q_1') - \mathfrak{P}(q_1)$ is equivalent to knowing $\mathfrak{P}(q_1')$.)

The maps $(i_{u,q_1,s})_*$ and $\mathbf{i}_{u,q_1,s}$. Since $i_{u,q_1,s} : \mathcal{L}_{ext}(q_1) \to \mathcal{H}_+(uq_1) \times W^+(uq_1)$ is a polynomial map, by the universal property of the tensor product, there exists $a > 0$ and a *linear* map $(i_{u,q_1,s})_* : \mathcal{L}_{ext}(q_1)^{\uplus a} \to \mathcal{H}_+(uq_1) \times W^+(uq_1)$ such that

$$i_{u,q_1,s} = (i_{u,q_1,s})_* \circ \hat{\mathbf{j}}.$$

Furthermore, there exists $r > a$ and a linear map $\mathbf{i}_{u,q_1,s} : \mathcal{L}_{ext}(q_1)^{\uplus r} \to \hat{\mathbf{H}}(uq_1)$ such that

$$\hat{\mathbf{j}} \circ i_{u,q_1,s} = \mathbf{i}_{u,q_1,s} \circ \hat{\mathbf{j}}. \tag{6.42}$$

Then $\mathbf{i}_{u,q_1,s}$ takes $F(q_1') - F(q_1) \in \mathcal{L}_{ext}(q_1)^{\uplus r}$ to $\hat{\mathbf{j}}(M_s, v_s) \in \hat{\mathbf{H}}(uq_1)$, where (M_s, v_s) is a parametrization of the approximation $\tilde{P}_s(uq_1; q_1')^{-1}U^+[q_1']$ to $U^+[q_1']$.

Construction of the map $\mathcal{A}(q_1, u, \ell, t)$. Recall that $\mathcal{B}$ denotes a "unit box" in U^+. Suppose $u \in \mathcal{B}$ and $q_1 \in X$. Let $s \in \mathbb{N}$ be a sufficiently large integer to be chosen later. (It will be chosen in Lemma 6.7, depending only on the Lyapunov spectrum). Let $r \in \mathbb{N}$ be such that (6.42) holds. Suppose $q_1 \in X$ and $u \in \mathcal{B}$. For $\ell > 0$ and $t > 0$, let

$$\mathcal{A}(q_1, u, \ell, t) : \mathcal{L}_{ext}(g_{-\ell}q_1)^{(r)} \to \mathbf{H}(g_t uq_1),$$

be given by

$$\mathcal{A}(q, u, \ell, t) = (g_t)_* \circ \mathbf{S}_{uq_1}^{Z(uq_1)} \circ \hat{\pi} \circ \mathbf{i}_{u,q_1,s} \circ (g_\ell)_*^{\uplus r}$$

where $(g_\ell)_* : \mathcal{L}_{ext}(q) \to \mathcal{L}_{ext}(g_\ell q)$ is given by

$$(g_\ell)_*(P) = g_\ell \circ P \circ g_\ell^{-1}.$$

Then $\mathcal{A}(q_1, u, \ell, t)$ is a *linear* map. Unraveling the definitions, we have, for $P \in \mathcal{L}_{ext}(g_{-\ell}q_1)$,

$$\mathcal{A}(q_1, u, \ell, t)(j^{\uplus r}(P)) = \mathbf{j}(G_t^+ \circ S_{uq_1}^{Z(uq_1)} \circ (i_{u,q_1,s}) \circ (g_\ell)_*(P)) \tag{6.43}$$

Thus,

$$\mathcal{A}(q_1, u, \ell, t)(F(q) - F(q')) = \mathbf{j}(M'', v''), \tag{6.44}$$

where $(M'', v'') \in \mathcal{H}_+(g_t uq_1) \times W^+(uq_1)$ is a parametrization of the approximation $g_t\tilde{P}_s(uq_1, u'q_1')^{-1}U^+[u'q_1']$ to $U^+[g_t u'q_1']$, where $u'q_1' \in U^+[q_1']$ is such that $d(g_t uq_1, g_t u'q_1') < 1$.

6.3*. Proofs of Lemma 6.7 and Lemma 6.8.

Proof of Lemma 6.7. Let $P = \mathfrak{P}(q') - \mathfrak{P}(q) \in \mathcal{L}_{ext}(q)$. Let

$$P_1 = (g_\ell)_*(P) = g_\ell \circ P \circ g_\ell^{-1} \in \mathcal{L}_{ext}(q_1).$$

Let

$$(M_s, v_s) = i_{u,q_1,s}(P_1).$$

Let $\widetilde{\mathcal{U}}_s = \widetilde{\mathcal{U}}_s(M_s, v_s)$ be the generalized affine subspace parametrized by (M_s, v_s). Then

$$\widetilde{\mathcal{U}}_s = \tilde{P}_s(uq_1, q_1')^{-1}U^+[q_1']. \tag{6.45}$$

Let

$$\widehat{\mathcal{U}} = \hat{P}(uq_1, q_1')^{-1}U^+[q_1'], \qquad \widehat{\mathcal{U}}_s = \hat{P}_s(uq_1, q_1')^{-1}U^+[q_1]. \tag{6.46}$$

By Lemma 6.12,

$$hd_{g_t uq_1}(\widehat{\mathcal{U}}, U^+[g_t u' q_1']) = O_{uq_1}(e^{-\alpha_1 t}), \tag{6.47}$$

where α_1 depends only on the Lyapunov spectrum. We have, in view of (5.6) and (5.7), for ℓ sufficiently large depending on δ,

$$\|P^-(q_1, q_1') - I\|_Y = O_{q_1}(e^{-\alpha_2 \ell}) \tag{6.48}$$

where α_2 depends only on the Lyapunov spectrum. Therefore,

$$hd_{uq_1}(U^+[uq_1], U^+[u'q_1']) = O_{q_1}(e^{-\alpha_2 \ell})$$

Since by the multiplicative ergodic theorem, the restriction of g_t to W^+ expands by a factor of at least $e^{\lambda_{min} t}$ in all directions, (for sufficiently large ℓ depending on δ), the assumption (6.18) implies

$$t < \alpha_3 \ell + c(uq_1) \tag{6.49}$$

where α_3 depends only on the Lyapunov spectrum.

To go from $\hat{Q}$ to $\hat{Q}_s$ we are doing order s Taylor expansion of the solution to (6.38) in the entries of $P^-(q_1, q_1') - I$. Thus, by (6.48),

$$\|\hat{Q}_s(uq_1; q_1') - \hat{Q}(uq_1; q_1')\|_Y = O_{q_1, uq_1}(e^{-\alpha_2 (s+1)\ell})$$

and thus, by (6.40),

$$\|\hat{P}_s(uq_1, q_1') - \hat{P}(uq_1, q_1')\|_Y = O_{q_1, uq_1}(e^{-\alpha_2 (s+1)\ell})$$

Then, by (6.46),

$$hd_{uq_1}(\widehat{\mathcal{U}}, \widehat{\mathcal{U}}_s) = O_{q_1, uq_1}(e^{-\alpha_2 (s+1)\ell}).$$

Then, by Lemma 6.6,

$$hd_{g_t uq_1}(g_t \widehat{\mathcal{U}}, g_t \widehat{\mathcal{U}}_s) = O_{q_1, uq_1}(e^{-\alpha_2 (s+1)\ell + 3t}). \tag{6.50}$$

Also, by (6.48), (6.36) and (6.34), we have

$$\|\hat{Q}(uq_1; q_1') - I\|_Y = O_{q_1, uq_1}(e^{-\alpha_2 \ell}),$$

and therefore

$$\|\hat{Q}_s(uq_1; q_1') - I\|_Y = O_{q_1, uq_1}(e^{-\alpha_2 \ell}),$$

Thus,

$$\|D_s\|_Y = \|\tilde{Q}_s(uq_1; q_1') - \hat{Q}_s(uq_1; q_1')\|_Y = O_{q_1}(e^{-\alpha_2 \ell})$$

Therefore, since D_s preserves all the eigenspaces $\mathcal{V}_i$, and the Osceledec multiplicative ergodic theorem, for sufficiently small $\epsilon > 0$ (depending on the Lyapunov spectrum),

$$\|g_t \circ D_s \circ g_t^{-1}\|_Y \le C_1(q_1) C_2(uq_1, \epsilon) e^{-\alpha_2 \ell + \epsilon t} \le C_1(q_1) C_2'(uq_1) e^{-(\alpha_2/2)\ell}.$$

Thus, by (6.45) and (6.46),

$$hd_{g_t uq_1}(g_t \widehat{\mathcal{U}}_s, g_t \widetilde{\mathcal{U}}_s) = O_{uq_1}(\|g_t \circ D_s \circ g_t^{-1}\|_Y) = O_{uq_1}(e^{-(\alpha_2/2)\ell}). \tag{6.51}$$

We now choose s so that $\alpha_2\alpha_3(s+1)-3>\alpha_2$. Then, by (6.49), (6.47), (6.50), and (6.51),

$$hd_{g_t uq_1}(g_t\widetilde{\mathcal{U}}_s, U^+[g_t u' q_1']) \le C(q_1)C(uq_1)e^{-\alpha\ell},$$

where α depends only on the Lyapunov spectrum. In view of (6.44), the pair (M'', v'') parametrizes $g_t\widetilde{\mathcal{U}}_s$. Therefore, (6.20) holds. Finally, (6.21) holds in view of Lemma 6.6 (b). $\square$

Proof of Lemma 6.8. Let $A_0 = g_{-t}A_t$, $A_0' = g_{-t}A_t'$. Let $\tilde{P}_s$ be as in (6.40). Let $\tilde{A}_t = \tilde{P}_s(g_t u' q_1', g_t uq_1)A_t'$. Then,

$$\tilde{A}_0 \equiv g_{-t}\tilde{A}_t = \tilde{P}_s(u'q_1', uq_1)A_0'.$$

As in the proof of Lemma 6.7, we have

$$\|\tilde{P}_s(u'q_1', uq_1) - I\|_Y = O(e^{-\alpha\ell}), \qquad \|\tilde{P}_s(g_t u'q_1', g_t uq_1) - I\|_Y = O(e^{-\alpha\ell}).$$

Hence, $|\tilde{A}_t|$ is comparable to $|A_t'|$ and $|\tilde{A}_0|$ is comparable to $|A_0'|$. Thus, it it enough to show that $|\tilde{A}_0|$ is comparable to $|A_0|$.

As in the proof of Lemma 6.7, let (M'', v'') be the pair parametrizing $g_t\tilde{\mathcal{U}}_s = \tilde{P}_s(g_t u' q_1, g_t uq_1)U^+[g_t u' q_1']$. Let $\tilde{f}_t : \mathrm{Lie}(U^+)(g_t uq_1) \to g_t\tilde{\mathcal{U}}_s$ be the "parametrization" map

$$\tilde{f}_t(Y) = \exp[(I+M'')Y](g_t uq_1)(g_t uq_1 + v'').$$

Similarly, let $f_t : \mathrm{Lie}(U^+)(g_t uq_1) \to U^+[g_t uq_1]$ be the exponential map

$$f_t(Y) = \exp(Y)g_t uq_1.$$

Then, provided that ϵ is sufficiently small, we have

$$0.5 f^{-1}(A_t) \subset \tilde{f}_t^{-1}(\tilde{A}_t) \subset 2 f^{-1}(A_t) \tag{6.52}$$

Let $M_0 = g_t^{-1}\circ M''\circ g_t$, $v_0 = g_t^{-1}v''$. Then, $g_t^{-1}\circ \tilde{f}_t \circ g_t = \tilde{f}_0$, where $\tilde{f}_0 : \mathrm{Lie}(U^+)(uq_1) \to \mathcal{U}_s$ is given by

$$\tilde{f}_0(Y) = \exp[(I+M_0)Y](g_t uq_1)(g_t uq_1 + v_0).$$

Similarly, $g_t^{-1}\circ f_t \circ g_t = f_0$, where $f_0 : \mathrm{Lie}(U^+)(uq_1) \to U^+[uq_1]$ is given by the exponential map

$$f_0(Y) = \exp(Y)uq_1.$$

Then, it follows from applying g_t^{-1} to (6.52) that

$$0.5 f_0^{-1}(A_0) \subset \tilde{f}_0^{-1}(\tilde{A}_0) \subset 2 f_0^{-1}(A_0) \tag{6.53}$$

Thus, $|\tilde{f}_0^{-1}(\tilde{A}_0)|$ is comparable to $|f_0^{-1}(A_0)| = |A_0|$. But, since $M'' \in \mathcal{H}_+(g_t uq_1)$ and $v'' \in W^+(g_t uq_1)$ are $O(\epsilon)$, M_0 and v_0 are exponentially small. Therefore, the map $\tilde{f}_0$ is close to f_0 (and since Y is small, it is close to the identity). Therefore, $|\tilde{f}_0^{-1}(\tilde{A}_0)|$ is comparable to $|\tilde{A}_0|$. The second assertion of the Lemma also follows from (6.53) and the fact that M_0 and v_0 are exponentially small. $\square$

7. Bilipshitz Estimates

Let $\|\cdot\|$ be the norm on $H^{(++)}_{big}$ defined in §4.5. Since $\mathbf{H} \subset H^{(++)}_{big}$, $\|\cdot\|$ is also a norm on $\mathbf{H}$. We can also define a norm on $H^{(--)}_{big}$ in an analogous way. Since $\mathcal{L}_{ext}(x)^{(r)} \subset H^{(--)}_{big}(x)$, the norm $\|\cdot\|_x$ is also a norm on $\mathcal{L}_{ext}(x)^{(r)}$. Let $A(q_1,u,\ell,t) = \|\mathcal{A}(q_1,u,\ell,t)\|$ where the operator norm is with respect to the dynamical norms $\|\cdot\|$ at $g_{-\ell q_1}$ and $g_t u q_1$. In the rest of this section we assume that on a set x of full measure, $\mathcal{L}(x) \not\subset U(x)$, and then by Proposition 6.9, $A(q_1,u,\ell,t) \to \infty$ as $t \to \infty$.

For $\epsilon > 0$, let

$$\hat{\tau}_{(\epsilon)}(q_1,u,\ell) = \sup\{t \ : \ t > 0 \text{ and } A(q_1,u,\ell,t) \le \epsilon.\}$$

Note that $\tau^H_{(\epsilon)}(q_1,u,0)$ need not be 0.

Let $\mathcal{A}_+(x,t) : \mathbf{H}(x) \to \mathbf{H}(g_t x)$ denote the action of g_t on $\mathbf{H}$ as in (6.15). Let $\mathcal{A}_-(q,s) : \mathcal{L}^{(r)}_{ext}(q) \to \mathcal{L}^{(r)}_{ext}(g_s q)$ denote the action of g_s on $\mathcal{L}^{(r)}_{ext}(q)$.

Lemma 7.1. *For all x, and $t > 0$,*

$$\|\mathcal{A}_-(q,t)\| \le e^{Nt}, \|\mathcal{A}_+(q,t)\| \le e^{Nt},$$

where N is an absolute constant. Also

$$\|\mathcal{A}_-(x,t)\| \ge e^{-\alpha t}, \tag{7.1}$$

and

$$\|\mathcal{A}_+(x,t)\| \ge e^{-\alpha t}, \tag{7.2}$$

Proof. This follows immediately from Proposition 4.11. □

Lemma 7.2. *Suppose $\epsilon > 0$. There exists $\kappa_1 > 1$ (depending only on the Lyapunov spectrum) with the following property: For all $s > 0$,*

$$\hat{\tau}_{(\epsilon)}(q_1,u,\ell+s) > \hat{\tau}_{(\epsilon)}(q_1,u,\ell) + \kappa_1^{-1} s.$$

Proof. Note that by (6.17),

$$\mathcal{A}(q_1,u,\ell+s,t+\tau) = \mathcal{A}_+(g_t u q_1,\tau)\mathcal{A}(q_1,u,\ell,t)\mathcal{A}_-(g_{-(\ell+s)}q_1,s).$$

Let $t = \hat{\tau}_{(\epsilon)}(q_1,u,\ell)$, so that $A(q_1,u,\ell+s,t+\tau) = \epsilon$. Therefore,

$$A(q_1,u,\ell+s,t+\tau) \le \|\mathcal{A}_+(q_t u q_1,\tau)\| A(q_1,u,\ell,t) \|\mathcal{A}_-(g_{-(\ell+s)}q_1,s)\| \le \\ \epsilon \|\mathcal{A}_+(q_t u q_1,\tau)\| \|\mathcal{A}_-(g_{-(\ell+s)}q_1,s)\| \le \epsilon e^{N\tau - \alpha s},$$

where we have used the fact that $A(q_1,u,\ell,t) = \epsilon$ and Lemma 7.1. If $t+\tau = \hat{\tau}_{(\epsilon)}(q_1,u,\ell+s)$ then $A(q_1,u,\ell+s,t+\tau) = \epsilon$. It follows that $N\tau - \alpha s > 0$, i.e. $\tau > (\alpha/N)s$. Hence,

$$\hat{\tau}_{(\epsilon)}(q_1,u,\ell+s) > \hat{\tau}_{(\epsilon)}(q_1,u,\ell) + (\alpha/N)s.$$

□

Lemma 7.3. *Suppose $\epsilon > 0$. There exists $\kappa_2 > 1$ (depending only on the Lyapunov spectrum) such that for all $s > 0$,*

$$\hat{\tau}_{(\epsilon)}(q_1, u, \ell + s) < \hat{\tau}_{(\epsilon)}(q_1, u, \ell) + \kappa_2 s.$$

Proof. We have

$$\mathcal{A}(q_1, u, \ell, t) = \mathcal{A}_+(g_{t+\tau} u q_1, -\tau)\mathcal{A}(q_1, u, \ell + s, t + \tau)\mathcal{A}_-(g_{-\ell} q_1, -s).$$

Let $t + \tau = \hat{\tau}_{(\epsilon)}(q_1, u, \ell + s)$. Then, by Lemma 7.1,

$$A(q_1, u, \ell, t) \leq \|\mathcal{A}_+(q_{t+\tau} u q_1, -\tau)\| A(q_1, u, \ell + s, t + \tau)\|\mathcal{A}_-(g_{-\ell} q_1, -s)\| \leq \\ \epsilon\|\mathcal{A}_+(q_{t+\tau} u q_1, -\tau)\|\|\mathcal{A}_-(g_{-\ell} q_1, -s)\| \leq \epsilon e^{-\alpha\tau + Ns},$$

where we have used the fact that $A(q_1, u, \ell + s, t + \tau) = \epsilon$. Since $A(q_1, u, \ell, t) = \epsilon$, it follows that $-\alpha\tau + Ns > 0$, i.e. $\tau < (N/\alpha)s$. It follows that

$$\hat{\tau}_{(\epsilon)}(q_1, u, \ell + s) < \hat{\tau}_{(\epsilon)}(q_1, u, \ell) + (N/\alpha)s$$

□

Proposition 7.4. *There exists $\kappa > 1$ depending only on the Lyapunov spectrum, and such that for $q_1 \in X$, $u \in \mathcal{B}$, any $\ell > 0$ and any subset $E_{bad} \subset \mathbb{R}^+$,*

$$|\hat{\tau}_{(\epsilon)}(q_1, u, E_{bad}) \cap [0, \ell]| \leq \kappa |E_{bad} \cap [0, \ell]| \tag{7.3}$$

$$|\{t \in [0, \ell] \ : \ \hat{\tau}_{(\epsilon)}(q_1, u, t) \in E_{bad}\}| \leq \kappa |E_{bad} \cap [0, \ell]|. \tag{7.4}$$

Proof. Let $\kappa = \max(\kappa_1^{-1}, \kappa_2)$, where κ_1, κ_2 are as in Lemma 7.2 and Lemma 7.3. Then, for fixed q_1, u, $\hat{\tau}_{(\epsilon)}(q_1, u, \ell)$ is κ-bilipshitz as a function of ℓ. The proposition follows immediately. □

We also note the following trivial lemma:

Lemma 7.5. *Suppose P and P' are subsets of $\mathbb{R}^n$, and we have*

$$P = \bigsqcup_{j=1}^{N} P_j \quad P' = \bigsqcup_{j=1}^{N} P'_j$$

Also suppose $Q \subset P$ and $Q' \subset P'$ are subsets with $|Q| > (1-\delta)|P|$, $|Q'| > (1-\delta)|P'|$, and also $|P'| = |P|$. Suppose for all $1 \leq j \leq N$ such that $P_j \cap Q \neq \emptyset$. $0.5|P_j| \leq |P'_j| \leq 2|P_j|$. Then there exists $\hat{Q} \subset Q$ with $|\hat{Q}| \geq (1 - 4\delta)|P|$ such that if j is such that $\hat{Q} \cap P_j \neq \emptyset$, then $Q' \cap P'_j \neq \emptyset$.

Proof. Let $J = \{j \;:\; P_j \cap Q \neq \emptyset\}$, and let $J' = \{j \;:\; Q' \cap P'_j \neq \emptyset\}$, and let

$$\hat{Q} = \bigsqcup_{j \in J \cap J'} Q \cap P_j.$$

Then,

$$|Q \setminus \hat{Q}| \le \sum_{j \in J/J'} |Q \cap P_j| \le \sum_{j \in J/J'} |P_j| = 2 \sum_{j \notin J'} |P'_j| \le 2|(Q')^c|,$$

since if $j \notin J'$ then $P'_j \subset (Q')^c$. Thus, $|Q \setminus \hat{Q}| \le 2\delta|P|$, and so $|\hat{Q}| \ge (1 - 4\delta)|P|$. $\square$

8. Preliminary Divergence Estimates

Motivation. Suppose in the notation of §2.3, q_1 and q'_1 are fixed, but $u \in \mathcal{B}$ varies. Then, as u varies, so do the points q_2 and q'_2, and thus the subspaces $U^+[q_2]$ and $U^+[q'_2]$. Let $\mathcal{U} = \mathcal{U}(M''(u), v''(u))$ be the approximation to $U^+[q'_2]$ given by Lemma 6.7, and as in Lemma 6.7, let $\mathbf{v}(u) = \mathbf{j}(M''(u), v''(u)) \in \mathbf{H}(q_2)$ be the associated vector in $\mathbf{H}(q_2)$.

In this section we define a certain g_t-equivariant and $(u)_*$-equivariant subbundle $\mathbf{E} \subset \mathbf{H}$ such that, for fixed q_1, q'_1, for most $u \in U^+[q_1]$, $\mathbf{v} = \mathbf{v}(u)$ is near $\mathbf{E}(q_2)$ (see Proposition 8.5 (a) below for the precise statement). The subbunde $\mathbf{E}$ is the direct sum of subbundles $\mathbf{E}_i$, where $\mathbf{E}_i$ is contained in the i-th Lyapunov subspace of $\mathbf{H}$, and also each $\mathbf{E}_i$ is both g_t-equivariant and $(u)_*$-equivariant.

8.1. **The subspaces $\mathbf{E}(x)$.** We apply the Osceledec multiplicative ergodic theorem to the action on $\mathbf{H}(x)$ (see (6.15)). We often drop the $*$ and denote the action simply by g_t.

Let

$$\{0\} = \mathbf{V}_0(x) \subset \mathbf{V}_1(x) \subset \cdots \subset \mathbf{V}_n(x) = \mathbf{H}(x)$$

denote the forward flag, and let

$$\{0\} = \hat{\mathbf{V}}_0(x) \subset \hat{\mathbf{V}}_1(x) \subset \cdots \subset \hat{\mathbf{V}}_n(x) = \mathbf{H}(x).$$

denote the backward flag. This means that almost all x and for $\mathbf{v} \in \mathbf{V}_i(x)$ such that $\mathbf{v} \notin \mathbf{V}_{i-1}(x)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\|g_{-t}\mathbf{v}\|}{\|\mathbf{v}\|} = \lambda_i, \tag{8.1}$$

and for $\mathbf{v} \in \hat{\mathbf{V}}_{n+1-i}(x)$ such that $\mathbf{v} \notin \hat{\mathbf{V}}_{n-i}(x)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\|g_t\mathbf{v}\|}{\|\mathbf{v}\|} = \lambda_i \tag{8.2}$$

By e.g. [GM, Lemma 1.5], we have for a.e. x,

$$\mathbf{H}(x) = \mathbf{V}_i(x) \oplus \hat{\mathbf{V}}_{n-i}(x) \tag{8.3}$$

Let

$$\mathbf{F}_j(x) = \{\mathbf{v} \in \mathbf{H}(x) \ : \ \text{for almost all } u \in \mathcal{B}(x),\ (u)_*\mathbf{v} \in \hat{\mathbf{V}}_{n-j+1}(ux)\}, \tag{8.4}$$

where $(u)_*$ is as in Lemma 6.2. In other words, if $\mathbf{v} \in \mathbf{F}_j(x)$, then for almost all $u \in U^+(x)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \|(g_t)_*(u)_*\mathbf{v}\| \le \lambda_j. \tag{8.5}$$

From the definition of $\mathbf{F}_j(x)$, we have

$$\{0\} = \mathbf{F}_{n+1}(x) \subseteq \mathbf{F}_n(x) \subseteq \mathbf{F}_{n-1}(x) \subseteq \dots \mathbf{F}_2(x) \subseteq \mathbf{F}_1(x) = \mathbf{H}(x). \tag{8.6}$$

Let

$$\mathbf{E}_j(x) = \mathbf{F}_j(x) \cap \mathbf{V}_j(x).$$

In particular, $\mathbf{E}_1(x) = \mathbf{V}_1(x)$. We may have $\mathbf{E}_j(x) = \{0\}$ if $j \neq 1$.

Lemma 8.1. *For almost all $x \in X$ the following holds: suppose $\mathbf{v} \in \mathbf{E}_j(x)$. Then for almost all $u \in \mathcal{B}(x)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|(g_t)_*(u)_*(\mathbf{v})\| = \lambda_j. \tag{8.7}$$

Thus, (recalling that $\mathcal{V}_j(\mathbf{H})$ denotes the subspace of $\mathbf{H}$ corresponding to the Lyapunov exponent λ_j), we have

$$\mathbf{E}_j(x) \subset \mathcal{V}_j(\mathbf{H})(x).$$

In particular, if $i \neq j$, $\mathbf{E}_i(x) \cap \mathbf{E}_j(x) = \{0\}$ for almost all $x \in X$.

Proof. Suppose $\mathbf{v} \in \mathbf{E}_j(x)$. Then $\mathbf{v} \in \mathbf{V}_j(x)$. Since in view of (8.1), $\mathbf{V}_j(ux) = (u)_*\mathbf{V}_j(x)$ for all $u \in U^+(x)$, we have for almost all $u \in \mathcal{B}(x)$, $\mathbf{v} \in \mathbf{V}_j(ux)$. It follows from (8.3) that (outside of a set of measure 0), $\mathbf{v} \notin \hat{\mathbf{V}}_{n-j}(ux)$. Now (8.7) follows from (8.2). □

Lemma 8.2. *After possibly modifying $\mathbf{E}_j(x)$ and $\mathbf{F}_j(x)$ on a subset of measure 0 of X, the following hold:*

(a) *$\mathbf{E}_j(x)$ and $\mathbf{F}_j(x)$ are g_t-equivariant, i.e. $(g_t)_*\mathbf{E}_j(x) = \mathbf{E}_j(g_t x)$, and $(g_t)_*\mathbf{F}_j(x) = \mathbf{F}_j(g_t x)$.*
(b) *For almost all $u \in U^+(x)$, $\mathbf{E}_j(ux) = (u)_*\mathbf{E}_j(x)$, and $\mathbf{F}_j(ux) = (u)_*\mathbf{F}_j(x)$.*

Proof. Note that for $t > 0$, $g_t\mathcal{B}[x] \supset \mathcal{B}[g_t x]$. Therefore, (a) for the case $t > 0$ follows immediately from the definitions of $\mathbf{E}_j(x)$ and $\mathbf{F}_j(x)$. Since the flow $\{g_t\}_{t>0}$ is ergodic, it follows that almost everywhere (8.4) holds with $\mathcal{B}[x]$ replaced by arbitrary large balls in $U^+[x]$. This implies that almost everywhere,

$$\mathbf{F}_j(x) = \{\mathbf{v} \in \mathbf{H}(x) \ : \ \text{for almost all } u \in U^+,\ (u)_*\mathbf{v} \in \hat{\mathbf{V}}_{n-j+1}(ux)\},$$

where $(u)_*\mathbf{v}$ is as in Lemma 6.2. Therefore (b) holds. Then, (a) for $t < 0$ also holds. □

Lemma 8.3. *Suppose there exists a subset $\Omega \subset X$ with $\nu(\Omega) = 1$ with the following property: suppose $x \in \Omega$, and $\mathbf{v} \in \mathbf{H}(x)$. Let*

$$Q(\mathbf{v}) = \{u \in \mathcal{B}(x) \;\; : \;\; (u)_* \mathbf{v} \in \hat{\mathbf{V}}_{n-j+1}(ux).\} \tag{8.8}$$

Then either $|Q(\mathbf{v})| = 0$, or $|Q(\mathbf{v})| = |\mathcal{B}(x)|$ (and thus $\mathbf{v} \in \mathbf{F}_j(x)$).

Proof. For a subspace $\mathbf{V} \subset \mathbf{H}(x)$, let

$$Q(\mathbf{V}) = \{u \in \mathcal{B}(x) \;\; : \;\; (u)_* \mathbf{V} \subset \hat{\mathbf{V}}_{n-j+1}(ux).\}$$

Let d be the maximal number such that there exists E' with $\nu(E') > 0$ such that for $x \in E'$ there exists a subspace $\mathbf{V} \subset \mathbf{H}(x)$ of dimension d with $|Q(\mathbf{V})| > 0$. For a fixed $x \in E'$, let $\mathcal{W}(x)$ denote the set of subspaces $\mathbf{V}$ of dimension d for which $|Q(\mathbf{V})| > 0$. Then, by the maximality of d, if $\mathbf{V}$ and $\mathbf{V}'$ are distinct elements of $\mathcal{W}(x)$ then $Q(\mathbf{V}) \cap Q(\mathbf{V}') = \emptyset$. Let $\mathbf{V}_x \in \mathcal{W}(x)$ be such that $|Q(\mathbf{V}_x)|$ is maximal (among elements of $\mathcal{W}(x)$).

Let $\epsilon > 0$ be arbitrary, and suppose $x \in E'$. By Lemma 3.11, there exists $t_0 > 0$ and a subset $Q(\mathbf{V}_x)^* \subset Q(\mathbf{V}_x) \subset \mathcal{B}(x)$ such that for all $u \in Q(\mathbf{V}_x)^*$ and all $t > t_0$,

$$|\mathcal{B}_t(ux) \cap Q(\mathbf{V}_x)| \geq (1-\epsilon)|\mathcal{B}_t(ux)|. \tag{8.9}$$

(In other words, $Q(\mathbf{V}_x)^*$ are "points of density" for $Q(\mathbf{V}_x)$, relative to the "balls" $\mathcal{B}_t$.) Let

$$E^* = \{ux \;\; : \;\; x \in E', \quad u \in Q(\mathbf{V}_x)^*\}.$$

Then, $\nu(E^*) > 0$. Let $\Omega = \{x \in X \;\; : \;\; g_{-t}x \in E^*\}$ infinitely often. Then $\nu(\Omega) = 1$. Suppose $x \in \Omega$. We can choose $t > t_0$ such that $g_{-t}x \in E^*$. Note that

$$\mathcal{B}[x] = g_t \mathcal{B}_t[g_{-t}x]. \tag{8.10}$$

Let $x' = g_{-t}x$, and let $\mathbf{V}_{t,x} = (g_t)_* \mathbf{V}_{x'}$. Then in view of (8.9) and (8.10),

$$|Q(\mathbf{V}_{t,x})| \geq (1-\epsilon)|\mathcal{B}(x)| \tag{8.11}$$

By the maximality of d (and assuming $\epsilon < 1/2$), $\mathbf{V}_{t,x}$ does not depend on t. Hence, for every $x \in \Omega$, there exists $\mathbf{V} \subset \mathbf{H}(x)$ such that $\dim \mathbf{V} = d$ and $|Q(\mathbf{V})| \geq (1-\epsilon)|\mathcal{B}(x)|$. Since $\epsilon > 0$ is arbitrary, for each $x \in \Omega$, there exists $\mathbf{V} \subset \mathbf{H}(x)$ with $\dim \mathbf{V} = d$, and $|Q(\mathbf{V})| = |\mathcal{B}(x)|$. Now the maximality of d implies that if $\mathbf{v} \notin \mathbf{V}$ then $|Q(\mathbf{v})| = 0$. $\square$

By Lemma 8.1, $\mathbf{E}_j(x) \cap \mathbf{E}_k(x) = \{0\}$ if $j \neq k$. Let

$$\Lambda' = \{i \;\; : \;\; \mathbf{E}_i \neq \{0\}\}.$$

Let

$$\mathbf{E}(x) = \bigoplus_{i \in \Lambda'} \mathbf{E}_i(x).$$

Then $\mathbf{E}(x) \subset \mathbf{H}(x)$.

In view of (8.5), (8.6) and Lemma 8.1, we have $\mathbf{F}_j(x) = \mathbf{F}_{j+1}(x)$ unless $j \in \Lambda'$. Therefore if we write the elements of Λ' in decreasing order as $i_1, \dots, i_m$ we have the flag (consisting of distinct subspaces)

$$\{0\} = \mathbf{F}_{i_{n+1}} \subset \mathbf{F}_{i_n}(x) \subset \mathbf{F}_{i_{n-1}}(x) \subset \dots \mathbf{F}_{i_2}(x) \subset \mathbf{F}_{i_1}(x) = \mathbf{H}(x). \tag{8.12}$$

For each $x \in X$, and $1 \leq r \leq n$, let $\mathbf{F}'_{i_r}(x)$ be the orthogonal complement (using the inner product $\langle \cdot, \cdot \rangle_x$ defined in §4.5) to $\mathbf{F}_{i_{r+1}}(x)$ in $\mathbf{F}_{i_r}(x)$.

Lemma 8.4. *There exists a compact $K_{01} \subset X$ with $\nu(K_{01}) > 1 - \delta$, $\beta(\delta) > 0$, and for every $x \in K_{01}$ a subset $Q_{01} \subset \mathcal{B}(x)$ with $|Q_{01}| > (1-\delta)|\mathcal{B}(x)|$ such that for any $j \in \Lambda'$ any $\mathbf{v}' \in \mathbf{F}'_j(x)$ and any $u \in Q_{01}(x)$, we can write*

$$(u)_* \mathbf{v}' = \mathbf{v}_u + \mathbf{w}_u, \quad \mathbf{v}_u \in \mathbf{E}_j(ux), \quad \mathbf{w}_u \in \hat{\mathbf{V}}_{n-j}(ux), \tag{8.13}$$

with $\|\mathbf{v}_u\| \geq \beta(\delta)\|\mathbf{v}'\|$.

Proof. This is a corollary of Lemma 8.3. Let $\Phi \subset X$ be the conull set where (8.3) holds and where $\mathbf{F}_i(x) = \mathbf{F}_{i+1}(x)$ for all $i \notin \Lambda'$. Suppose $x \in \Phi$.

Let $\mathbf{F}_k(x) \subset \mathbf{F}_j(x)$ be the next subspace in the flag (8.12), (i.e. $\mathbf{F}_k = \{0\}$ if j is the maximal index in Λ' and otherwise we have $k > j$ be minimal such that $k \in \Lambda'$.) Then $\mathbf{F}_{j+1}(x) = \mathbf{F}_k(x)$. Since $\mathbf{F}'_j(x)$ is complimentary to $\mathbf{F}_k(x)$ we have that $\mathbf{F}'_j(x)$ is complementary to $\mathbf{F}_{j+1}(x)$.

By Lemma 8.2, $\mathbf{F}_j$ is g_t-equivariant, and therefore, by the multiplicative ergodic theorem applied to $\mathbf{F}_j$, $\mathbf{F}_j$ is the direct sum of its Lyapunov subspaces. Therefore, in view of (8.3), for almost all $y \in X$,

$$\mathbf{F}_j(y) = (\mathbf{F}_j(y) \cap \mathbf{V}_j(y)) \oplus (\mathbf{F}_j(y) \cap \hat{\mathbf{V}}_{n-j}(y)). \tag{8.14}$$

Since $\mathbf{F}'_j(x) \subset \mathbf{F}_j(x)$, we have by Lemma 8.2, $(u)_*\mathbf{v}' \in \mathbf{F}_j(ux)$ for almost all $u \in \mathcal{B}$. By the definition of $\mathbf{F}_{j+1}(x)$, since $\mathbf{v}' \notin \mathbf{F}_{j+1}(x)$, for almost all u if we decompose using (8.14),

$$(u)_* \mathbf{v}' = \mathbf{v}_u + \mathbf{w}_u, \quad \mathbf{v}_u \in \mathbf{F}_j(ux) \cap \mathbf{V}_j(ux), \quad \mathbf{w}_u \in \mathbf{F}_j(ux) \cap \hat{\mathbf{V}}_{n-j}(ux),$$

then $\mathbf{v}_u \neq 0$. Since by definition $\mathbf{F}_j(x) \cap \mathbf{V}_j(x) = \mathbf{E}_j(x)$ we have $\mathbf{v}_u \in \mathbf{E}_j(x)$. Let

$$E_n(x) = \{\mathbf{v}' \in \mathbb{P}^1(\mathbf{F}'(x)) \ : \ |\{u \in \mathcal{B}(x) \ : \ \|\mathbf{v}_u\| \geq \tfrac{1}{n}\|\mathbf{v}'\|\}| > (1-\delta)|\mathcal{B}(x)|\}.$$

Then the $E_n(x)$ are an increasing family of open sets, and $\bigcup_{n=1}^{\infty} E_n(x) = \mathbb{P}^1(\mathbf{F}'_j(x))$. Since $\mathbb{P}^1(\mathbf{F}'_j(x))$ is compact, there exists $n(x)$ such that $E_{n(x)}(x) = \mathbb{P}^1(\mathbf{F}'_j(x))$. We can now choose $K_{01} \subset \Phi$ with $\nu(K_{01}) > 1 - \delta$ such that for $x \in K_{01}$, $n(x) < 1/\beta(\delta)$. $\square$

Proposition 8.5. *For every $\delta > 0$ there exists a subset K of measure at least $1-\delta$ and a number $L_2(\delta) > 0$ such that the following holds: Suppose $x \in \pi^{-1}(K)$, $\mathbf{v} \in \mathbf{H}(x)$. Then,*

(a) *For any $L' > L_2(\delta)$ there exists $L' < t < 2L'$ such that for at least $(1-\delta)$-fraction of $u \in \mathcal{B}(g_{-t}x)$,*

$$d\left(\frac{(g_s)_*(u)_*(g_{-t})_*\mathbf{v}}{\|(g_s)_*(u)_*(g_{-t})_*\mathbf{v}\|}, \mathbf{E}(g_s u g_{-t} x)\right) \le C(\delta)e^{-\alpha t},$$

where $s > 0$ is such that

$$\|(g_s)_*(u)_*(g_{-t})_*\mathbf{v}\| = \|\mathbf{v}\|, \tag{8.15}$$

and α depends only on the Lyapunov spectrum.

(b) *There exists $\epsilon' > 0$ (depending only on the Lyapunov spectrum) and for every $\delta > 0$ a compact set K'' with $\nu(K'') > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$ such such that the following holds: Suppose there exist arbitrarily large $t > 0$ with $g_{-t}x \in K''$ so that for at least $(1-\delta)$-fraction of $u \in \mathcal{B}(x)$, the number $s > 0$ satisfying (8.15), also satisfies*

$$s \ge (1-\epsilon')t. \tag{8.16}$$

Then $\mathbf{v} \in \mathbf{E}(x)$.

Proof. Let $\epsilon > 0$ be smaller then one third of the difference between any two Lyapunov exponents for the action on $\mathbf{H}$. By the Osceledec multiplicative ergodic theorem, there exists a compact subset $K_1 \subset X$ with $\nu(K_1) > 1 - \delta/2$ and $L > 0$ such that for $x \in K_1$ and all j and all $s > L$,

$$\|(g_t)_*\mathbf{v}\| \le e^{(\lambda_j+\epsilon)s}\|\mathbf{v}\|, \qquad \mathbf{v} \in \hat{\mathbf{V}}_{n-j+1}(x)$$

and

$$\|(g_t)_*\mathbf{v}\| \ge e^{(\lambda_j-\epsilon)s}\|\mathbf{v}\|, \qquad \mathbf{v} \in \mathbf{V}_j(x).$$

Let K_1^* be such that for $x \in K_1^*$,

$$|\{u \in \mathcal{B}(x) \;:\; ux \in K_1\}| \ge (1-\delta/2)|\mathcal{B}(x)|.$$

Let $K'' = K_{01} \cap K_1^*$, where K_{01} is as in Lemma 8.4. Let K, $L_2(\delta)$ be such that for all $x \in K$ and all $L' > L_2$, there exists t with $L' < t < 2L'$ and $g_{-t}x \in K''$. Write

$$(g_{-t})_*\,\mathbf{v} = \sum_{j\in\Lambda'} \mathbf{v}'_j, \qquad \mathbf{v}'_j \in \mathbf{F}'_j(g_{-t}x). \tag{8.17}$$

We have $g_{-t}x \in K_{01} \cap K_1^*$. Suppose $u \in Q_{01}$ and $ux \in K_1$. Then, by Lemma 8.4, we have

$$(u)_*(g_{-t})_*\,\mathbf{v} = \sum_{j\in\Lambda}(\mathbf{v}_j + \mathbf{w}_j), \tag{8.18}$$

where $\mathbf{v}_j \in \mathbf{E}_j(g_{-t}x)$, $\mathbf{w}_j \in \hat{\mathbf{V}}_{n-j}(g_{-t}x)$, and for all $j \in \Lambda'$,

$$\|\mathbf{v}_j\| \ge \beta(\delta)\|\mathbf{v}'_j\| \ge \beta'(\delta)\|\mathbf{w}_j\|. \tag{8.19}$$

Then,

$$\|(g_s)_*\mathbf{w}_j\| \le e^{(\lambda_{j+1}+\epsilon)s}\|\mathbf{w}_j\|,$$

and,

$$\|(g_s)_*\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\|\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\|\mathbf{w}_j\|. \tag{8.20}$$

Thus, for all $j \in \Lambda'$.

$$\|(g_s)_*\mathbf{w}_j\| \leq e^{-(\lambda_j-\lambda_{j+1}+2\epsilon)s}\|(g_s)_*\mathbf{v}_j\|.$$

Since $(g_s)_*\mathbf{v}_j \in \mathbf{E}$ and using part (a) of Proposition 4.11, we get (a) of Proposition 8.5.

To prove (b), suppose $\mathbf{v} \notin \mathbf{E}(x)$. We may write

$$\mathbf{v} = \sum_{i\in\Lambda'} \hat{\mathbf{v}}_i, \qquad \hat{\mathbf{v}}_i \in \mathbf{F}'_i(x)$$

Let j be minimal such that $\hat{\mathbf{v}}_j \notin \mathbf{E}_j(x)$. Let $k > j$ be such that $\mathbf{F}_k(x) \subset \mathbf{F}_j(x)$ is the subspace preceding $\mathbf{F}_j(x)$ in (8.12). Then, $\mathbf{F}_i(x) = \mathbf{F}_j(x)$ for $k+1 \leq i \leq j$.

Since $\hat{\mathbf{v}}_j \notin \mathbf{E}_j(x)$, $\hat{\mathbf{v}}_j$ must have a component in $\mathcal{V}_i(\mathbf{H})(x)$ for some $i \geq j+1$. Therefore, by looking only at the component in $\mathcal{V}_i(H)$, we get

$$\|(g_{-t})_*\mathbf{v}\| \geq C(\mathbf{v})e^{-(\lambda_{j+1}+\epsilon)t},$$

Also since $\mathbf{F}_k$ is g_t-equivariant, we have $\mathbf{F}_k(x) = \bigoplus_m \mathbf{F}_k(x) \cap \mathcal{V}_m(\mathbf{H})$. Therefore, (again by looking only at the component in $\mathcal{V}_i(\mathbf{H})$), we get

$$d((g_{-t})_*\mathbf{v}, \mathbf{F}_k(g_{-t}x)) \geq C(\mathbf{v})e^{-(\lambda_{j+1}+2\epsilon)t}.$$

Therefore, (since $(g_{-t})_*\mathbf{v} \in \mathbf{F}_j(x)$), we see that if we decompose $(g_{-t})_*\mathbf{v}$ as in (8.17), we get

$$\|\mathbf{v}'_j\| \geq C(\mathbf{v})e^{-(\lambda_{j+1}+2\epsilon)t},$$

We now decompose $(u)_*(g_{-t})_*\mathbf{v}$ as in (8.18). Then, from (8.19) and (8.20),

$$\|(g_s)_*\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\|\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)s}\beta(\delta)\|\mathbf{v}'_j\| \geq e^{(\lambda_j-\epsilon)s}\beta(\delta)C(\mathbf{v})e^{-(\lambda_{j+1}+2\epsilon)t}. \tag{8.21}$$

If s satisfies (8.15), then $\|(g_s)_*\mathbf{v}_j\| = O(1)$. Therefore, in view of (8.21),

$$e^{(\lambda_j-\epsilon)s}e^{-(\lambda_{j+1}+2\epsilon)t} \leq c = c(\mathbf{v},\delta).$$

Therefore,

$$s \leq \frac{(\lambda_{j+1}+2\epsilon)t + \log c(\mathbf{v},\delta)}{(\lambda_j-\epsilon)}.$$

Since $\lambda_j > \lambda_{j+1}$, this contradicts (8.16) if ϵ is sufficiently small and t is sufficently large. $\square$

9. The action of the cocycle on $\mathbf{E}$

9.1. **The Jordan canonical form of the cocycle on $\mathbf{E}(x)$.** We consider the action of the cocycle on $\mathbf{E}$. The Lyapunov exponents are λ_i, $i \in \Lambda'$. We note that by Lemma 8.2, the bundle $\mathbf{E}^+$ admits the equivariant measurable flat U^+-connection given by the maps $(u)_* : \mathbf{E}(x) \to \mathbf{E}(y)$, where $(u)_*$ is as in Lemma 6.2. This connection satisfies the condition (4.11), since by Lemma 8.2, $(u)_* \mathbf{E}_j(x) = \mathbf{E}_j(y)$. For each $i \in \tilde{\Lambda}$, we have the maximal flag as in Lemma 4.3,

$$\{0\} \subset \mathbf{E}_{i1}(x) \subset \cdots \subset \mathbf{E}_{i,n_i}(x) = \mathbf{E}_i(x). \tag{9.1}$$

Let Λ'' denote the set of pairs ij which appear in (9.1). By Proposition 4.4 and Remark 4.8, we have for $u \in \mathcal{B}(x)$,

$$(u)_* \mathbf{E}_{ij}(x) = \mathbf{E}_{ij}(ux).$$

Let $\| \cdot \|_x$ and $\langle \cdot, \cdot \rangle_x$ denote the restriction to $\mathbf{E}$ of the norm and inner product on $H^{(++)}_{big}(x)$ defined in §4.5. (We will often omit the subscript from $\langle \cdot, \cdot \rangle_x$ and $\| \cdot \|_x$.) Then, the distinct $\mathbf{E}_i(x)$ are orthogonal. For each $ij \in \Lambda''$ let $\mathbf{E}'_{ij}(x)$ be the orthogonal complement (relative to the inner product $\langle \cdot, \cdot \rangle_x$) to $\mathbf{E}_{i,j-1}(x)$ in $\mathbf{E}_{ij}(x)$.

Then, by Proposition 4.11, we can write, for $\mathbf{v} \in \mathbf{E}'_{ij}(x)$,

$$(g_t)_* \mathbf{v} = e^{\lambda_{ij}(x,t)} \mathbf{v}' + \mathbf{v}'', \tag{9.2}$$

where $\mathbf{v}' \in \mathbf{E}'_{ij}(g_t x)$, $\mathbf{v}'' \in \mathbf{E}_{i,j-1}(g_t x)$, and $\|\mathbf{v}'\| = \|\mathbf{v}\|$. Hence (since $\mathbf{v}'$ and $\mathbf{v}''$ are orthogonal),

$$\|(g_t)_* \mathbf{v}\| \geq e^{\lambda_{ij}(x,t)} \|\mathbf{v}\|. \tag{9.3}$$

In view of Proposition 4.11 there exists a constant $\kappa > 1$ such that for a.e $x \in X$ and for all $\mathbf{v} \in \mathbf{E}(x)$ and all $t \geq 0$,

$$e^{\kappa^{-1} t} \|\mathbf{v}\| \leq \|(g_t)_* \mathbf{v}\| \leq e^{\kappa t} \|\mathbf{v}\|. \tag{9.4}$$

Lemma 9.1. *For $y = ux \in \mathcal{B}[x]$, the connection $(u)_* : \mathbf{E}(x) \to \mathbf{E}(y)$ agrees with the restriction to $\mathbf{E}$ of the connection $\mathbf{P}^+(x,y)$ induced from the map $P^+(x,y)$ defined in §4.2.*

Proof. Let $V_i(x)$ and $\mathcal{V}_i(x)$ be as in §4.1. Consider the definition (6.10) of $(u)_*$ in §6. For a fixed $Y = \log u \in \mathrm{Lie}(U^+)(x)$ and $M \in \mathcal{H}_+(x)$, let $h : W^+(x) \to W^+(ux)$ be given by

$$h(v) = \exp((I+M)Y)(x+v) - \exp(Y)x.$$

From the form of h, we see that $h(V_i(x)) = V_i(ux)$, and also, h induces the identity map on $V_i(x)/V_{i-1}(x) = V_i(ux)/V_{i-1}(ux)$. Thus, for $v \in \mathcal{V}_i(x)$,

$$h(v) = P^+(x, ux) v + V_{i-1}(ux).$$

This means that for $\mathbf{v} \in \mathbf{E}_i(x)$,

$$(u)_* \mathbf{v} = \mathbf{P}^+(x, ux) \mathbf{v} + \mathbf{V}_{i-1}(\mathbf{H})(ux).$$

But, for $\mathbf{v} \in \mathbf{E}_i(x)$, $(u)_*\mathbf{v} \in \mathbf{E}_i(ux)$ (and thus has no component in $\mathbf{V}_{i-1}(\mathbf{H})(ux)$). Hence, for all $\mathbf{v} \in \mathbf{E}_i(x)$, we have $(u)_*\mathbf{v} = \mathbf{P}^+(x, ux)\mathbf{v}$. □

9.2. **Time changes.**

The flows g_t^{ij} and the time changes $\hat{\tau}_{ij}(x,t)$. We define the time changed flow g_t^{ij} so that (after the time change) the cocycle $\lambda_{ij}(x,t)$ of (9.2) becomes $\lambda_i t$. We write $g_t^{ij} x = g_{\hat{\tau}_{ij}(x,t)} x$. Then, by construction, $\lambda_{ij}(x, \hat{\tau}_{ij}(x,t)) = \lambda_i t$. We note the following:

Lemma 9.2. *Suppose $y \in W^+[x] \cap J[x]$. Then for any $ij \in \Lambda''$ and any $t > 0$,*

$$g_{-t}^{ij} y \in W^+[g_{-t}^{ij} x] \cap J[g_{-t}^{ij} x].$$

Proof. This follows immediately from property (e) of Proposition 3.6, and the definition of the flow g_{-t}^{ij}. □

In view of Proposition 4.11, we have

$$\frac{1}{\kappa}|t - t'| \le |\hat{\tau}_{ij}(x,t) - \hat{\tau}_{ij}(x,t')| \le \kappa |t - t'| \tag{9.5}$$

where κ depends only on the Lyapunov spectrum.

The flows $g_t^{\mathbf{v}}$. Suppose $\mathbf{v} \in \mathbf{E}(x)$. Let $g_t^{\mathbf{v}} x = g_{\hat{\tau}_{\mathbf{v}}(x,t)} x$, where the time change $\hat{\tau}_{\mathbf{v}}(x,t)$ is chosen so that

$$\|(g_t^{\mathbf{v}})_* \mathbf{v}\|_{g_t^{\mathbf{v}} x} = e^t \|\mathbf{v}\|_x.$$

We transport the vector $\mathbf{v}$ along the flow using the action of $(g_t)_*$; this allows us to define $g_t^{\mathbf{v}}$ for all t. By (9.4), (9.5) holds for $\hat{\tau}_{\mathbf{v}}$ instead of $\hat{\tau}_{ij}$. Also, Lemma 9.2 still holds if g_t^{ij} is replaced by $g_t^{\mathbf{v}}$.

9.3. **The foliations $\mathcal{F}_{ij}$, $\mathcal{F}_{\mathbf{v}}$ and the parallel transport $R(x,y)$.** For $x \in \tilde{X}$, let

$$G[x] = \{ g_s u g_{-t} x \;\;:\;\; t \ge 0, s \ge 0, u \in \mathcal{B}(g_{-t}x) \} \subset \tilde{X}.$$

For $y = g_s u g_{-t} x \in G[x]$, let

$$R(x,y) = (g_s)_* (u)_* (g_{-t})_*.$$

Here $(g_s)_*$ is as in (6.15) and $(u)_* : \mathbf{E}(g_{-t}x) \to \mathbf{E}(ug_{-t}x)$ is as in Lemma 6.2. It is easy to see that $R(x,y)$ depends only on x, y and not on the choices of t, u, s.

In view of (9.2) and Lemma 9.1, we have, for $\mathbf{v} \in \mathbf{E}'_{ij}(x)$, and any $y = g_s u g_{-t} x \in G[x]$,

$$R(x,y)\mathbf{v} = e^{\lambda_{ij}(x,y)} \mathbf{v}' + \mathbf{v}'' \tag{9.6}$$

where $\mathbf{v}' \in \mathbf{E}'_{ij}(y)$, $\mathbf{v}'' \in \mathbf{E}_{i,j-1}(y)$, and $\|\mathbf{v}'\| = \|\mathbf{v}\|$. In (9.6), we have

$$\lambda_{ij}(x,y) = \lambda_{ij}(x,-t) + \lambda_{ij}(ug_{-t}x, s). \tag{9.7}$$

Notational convention. We sometimes use the notation $R(x,y)$ when $x \in X$ (instead of $\tilde{X}$) and $y \in G[x]$. Similarly, when $x \in X$, we can think of the leaf of the foliation $\mathcal{F}_{ij}[x]$ as a subset of X (not $\tilde{X}$).

For $x \in \tilde{X}$ and $ij \in \Lambda''$, let $\mathcal{F}_{ij}[x]$ denote the set of $y \in G[x]$ such that there exists $\ell \geq 0$ so that

$$g_{-\ell}^{ij} y \in \mathcal{B}[g_{-\ell}^{ij} x] \tag{9.8}$$

By Lemma 9.2, if (9.8) holds for some ℓ, it also holds for any bigger ℓ. Alternatively,

$$\mathcal{F}_{ij}[x] = \{g_{\ell}^{ij} u g_{-\ell}^{ij} x \;\;:\;\; \ell \geq 0, \quad u \in \mathcal{B}(g_{-\ell}^{ij} x)\} \subset \tilde{X}.$$

In view of (9.7), it follows that

$$\lambda_{ij}(x,y) = 0 \qquad \text{if } y \in \mathcal{F}_{ij}[x]. \tag{9.9}$$

We refer to the sets $\mathcal{F}_{ij}[x]$ as *leaves*. Locally, the leaf $\mathcal{F}_{ij}[x]$ through x is a piece of $U^+[x]$. More precisely, for $y \in \mathcal{F}_{ij}[x]$,

$$\mathcal{F}_{ij}[x] \cap J[y] \subset U^+[y].$$

Then, for any compact subset $A \subset \mathcal{F}_{ij}[x]$ there exists ℓ large enough so that $g_{-\ell}^{ij}(A)$ is contained in a set of the form $\mathcal{B}[z] \subset U^+[z]$. Then the same holds for $g_{-t}^{ij}(A)$, for any $t > \ell$.

Note that the sets $\mathcal{B}[x]$ support a "Lebesgue measure", namely the pushforward of the Haar measure on $U^+/Q_+(x)$ to $\mathcal{B}[x]$ under the map $u \to ux$. (Recall that $Q_+(x)$ is the stabilizer of x in the affine group $\mathcal{G}(x)$). As a consequence, the leaves $\mathcal{F}_{ij}[x]$ also support a Lebesgue measure (defined up to normalization), which we denote by $|\cdot|$. More precisely, if $A \subset \mathcal{F}_{ij}[x]$ and $B \subset \mathcal{F}_{ij}[x]$ are compact subsets, we define

$$\frac{|A|}{|B|} \equiv \frac{|g_{-\ell}^{ij}(A)|}{|g_{-\ell}^{ij}(B)|}, \tag{9.10}$$

where ℓ is chosen large enough so that both $g_{-\ell}^{ij}(A)$ and $g_{-\ell}^{ij}(B)$ are contained in a set of the form $\mathcal{B}[z]$, $z \in X$. It is clear that if we replace ℓ by a larger number, the right-hand-side of (9.10) remains the same.

We define the "balls" $\mathcal{F}_{ij}[x,\ell] \subset \mathcal{F}_{ij}[x]$ by

$$\mathcal{F}_{ij}[x,\ell] = \{y \in \mathcal{F}_{ij}[x] \;\;:\;\; g_{-\ell}^{ij} y \in \mathcal{B}[g_{-\ell}^{ij} x]\}. \tag{9.11}$$

Lemma 9.3. *Suppose $x \in \tilde{X}$ and $y \in \mathcal{F}_{ij}[x]$. Then, for ℓ large enough,*

$$\mathcal{F}_{ij}[x,\ell] = \mathcal{F}_{ij}[y,\ell].$$

Proof. Suppose $y \in \mathcal{F}_{ij}[x]$. Then, for ℓ large enough, $g_{-\ell}^{ij} y \in \mathcal{B}[g_{-\ell}^{ij} x]$, and then $\mathcal{B}[g_{-\ell}^{ij} y] = \mathcal{B}[g_{-\ell}^{ij} x]$. $\square$

The foliations $\mathcal{F}_{\mathbf{v}}$. For $\mathbf{v} \in \mathbf{E}(x)$ we can define the foliations $\mathcal{F}_{\mathbf{v}}[x]$ and the "balls" $\mathcal{F}_{\mathbf{v}}[x,\ell]$ as in (9.8) and (9.11), with $g_t^{\mathbf{v}}$ replacing the role of g_t^{ij}.

For $y \in \mathcal{F}_{\mathbf{v}}[x]$, we have

$$\mathcal{F}_{\mathbf{v}}[x] = \mathcal{F}_{\mathbf{w}}[y], \qquad \text{where } \mathbf{w} = R(x,y)\mathbf{v}.$$

For $y \in \mathcal{F}_{\mathbf{v}}[x]$ and $\ell > 0$, let

$$\tilde{g}_{-\ell}(y) = g^{\mathbf{w}}_{-\ell} y, \quad \text{where } \mathbf{w} = R(x,y)\mathbf{v}. \tag{9.12}$$

(When there is a potential for confusion about the foliation used, we denote $\tilde{g}_{-\ell}$ by $\tilde{g}^{\mathbf{v},x}_{-\ell}$.) We can define the measure (up to normalization) $|\cdot|$ on $\mathcal{F}_{\mathbf{v}}[x,\ell]$ as in (9.10). Lemma 9.3 holds for $\mathcal{F}_{\mathbf{v}}[x]$ without modifications.

10. Bounded Subspaces and synchronized exponents

Recall that Λ'' indexes the "fine Lyapunov spectrum" on $\mathbf{E}$. In this section we define an equivalence relation called "synchronization" on Λ''; the set of equivalence classes is denoted by $\tilde{\Lambda}$. For each $ij \in \Lambda''$ we define a g_t-equivariant and locally $(u)_*$-equivariant (in the sense of Lemma 6.2 (b)) subbundle $\mathbf{E}_{ij,bdd}$ of the bundle $\mathbf{E}_i \equiv \mathcal{V}_i(\mathbf{E})$ so that,

$$\mathbf{E}_{ij,bdd} \subset \mathbf{E}_{ik,bdd} \qquad \text{if } [ij] = [ik] \text{ and } j \le k.$$

For an equivalence class $[ij] \in \tilde{\Lambda}$ let $[ij]'$ denote the set of pairs kr such that $kr \in [ij]$ and $kr' \notin [ij]$ for all $r' > r$. Finally, let

$$\mathbf{E}_{[ij],bdd}(x) = \bigoplus_{kr \in [ij]'} \mathbf{E}_{kr,bdd}(x),$$

Then, we claim that the following three propositions hold:

Proposition 10.1. *There exists $\theta > 0$ depending only on ν such that the following holds: for every (sufficiently small depending on θ) $\delta > 0$ and every $\eta > 0$, there exists a subset $K = K(\delta,\eta)$ of measure at least $1-\delta$ and $L_0 = L_0(\delta,\eta) > 0$ such that the following holds: Suppose $\mathbf{v} \in \mathbf{E}(x)$, $L \ge L_0$, and*

$$|g_{[-1,1]}K \cap \mathcal{F}_{\mathbf{v}}[x,L]| \ge (1-(\theta/2)^{n+1})|\mathcal{F}_{\mathbf{v}}[x,L]|.$$

Then, for at least $(\theta/2)^n$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x,L]$ (where n depends only on the dimension),

$$d\left(\frac{R(x,y)\mathbf{v}}{\|R(x,y)\mathbf{v}\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < \eta.$$

Proposition 10.2. *There exists a function $C_3 : X \to \mathbb{R}^+$ finite almost everywhere so that for all $x \in \tilde{X}$, for all $y \in \mathcal{F}_{ij}[x]$, for all $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$,*

$$C_3(x)^{-1}C_3(y)^{-1}\|\mathbf{v}\| \le \|R(x,y)\mathbf{v}\| \le C_3(x)C_3(y)\|\mathbf{v}\|.$$

(Recall that by $C_3(x)$ we mean $C_3(\pi(x))$.)

Proposition 10.3. *There exists $\theta > 0$ (depending only on ν) and a subset $\Psi \subset X$ with $\nu(\Psi) = 1$ such that the following holds:*

Suppose $x \in \Psi$, $\mathbf{v} \in \mathbf{H}(x)$, and there exists $C > 0$ such that for all $\ell > 0$, and at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x, \ell]$,

$$\|R(x,y)\mathbf{v}\| \le C\|\mathbf{v}\|.$$

Then, $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$.

Proposition 10.1 is what allows us to choose u so that there exists u' such that the vector in $\mathbf{H}$ associated to the difference between the generalized subspaces $U^+[g_t u' q_1']$ and $U^+[g_t u q_1]$ points close to a controlled direction, i.e. close to $\mathbf{E}_{[ij],bdd}(g_t u q_1)$. This allows us to address "Technical Problem #3" from §2.3. Then, Proposition 10.2 and Proposition 10.3 are used in §11 to define and control conditional measures f_{ij} associated to each $[ij] \in \tilde{\Lambda}$, so we can implement the outline in §2.3. We note that it is important for us to define a family of subspaces so that all three propositions hold.

The number $\theta > 0$, the synchronization relation and the subspaces $\mathbf{E}_{ij,bdd}$ are defined in §10.1*. Also Proposition 10.1 is proved in §10.1*. Proposition 10.2 and Proposition 10.3 are proved in §10.2*. Both subsections may be skipped on first reading.

Example. To completely understand the example below, it necessary to read at least §10.1*. However, we include it here to give some flavor of the construction.

Suppose we have a basis $\{\mathbf{e}_1(x), \mathbf{e}_2(x), \mathbf{e}_3(x), \mathbf{e}_4(x)\}$ for $\mathbf{E}(x)$, relative to which the cocycle has the form (for $y \in G[x]$):

$$R(x,y) = \begin{pmatrix} e^{\lambda_{11}(x,y)} & u_{12}(x,y) & 0 & 0 \\ 0 & e^{\lambda_{12}(x,y)} & 0 & 0 \\ 0 & 0 & e^{\lambda_{31}(x,y)} & 0 \\ 0 & 0 & 0 & e^{\lambda_{41}(x,y)} \end{pmatrix}$$

Suppose $\mathbf{E}_1(x) = \mathbb{R}\mathbf{e}_1(x) \oplus \mathbb{R}\mathbf{e}_2(x)$ (so $\mathbf{e}_1$ and $\mathbf{e}_2$ correspond to the Lyapunov exponent λ_1), $\mathbf{E}_3(x) = \mathbb{R}\mathbf{e}_3(x)$, $\mathbf{E}_4(x) = \mathbb{R}\mathbf{e}_4(x)$ (so that $\mathbf{e}_3$ and $\mathbf{e}_4$ correspond to the Lyapunov exponents λ_3 and λ_4 respectively). Therefore the Lyapunov exponents λ_3 and λ_4 have multiplicity 1, while λ_1 has multiplicity 2.

Then, we have

$$\mathbf{E}_{31,bdd}(x) = \mathbb{R}\mathbf{e}_3(x), \qquad \mathbf{E}_{41,bdd}(x) = \mathbb{R}\mathbf{e}_4(x), \qquad \mathbf{E}_{11,bdd}(x) = \mathbb{R}\mathbf{e}_1(x).$$

(For example, if $y \in \mathcal{F}_{31}[x]$ then $\lambda_{31}(x,y) = 0$, so that by (9.6), $\|R(x,y)\mathbf{e}_3\| = \|\mathbf{e}_3\|$.)

Now suppose that 31 and 41 are synchronized, but all other pairs are not synchronized. (The see Definition 10.8 for the exact definition of synchronization, but roughly this means that $|\lambda_{41}(x,y)|$ is bounded as y varies over $\mathcal{F}_{31}[x]$, but for all other distinct pairs ij and kl, $|\lambda_{ij}(x,y)|$ is essentially unbounded as y varies over $\mathcal{F}_{kl}[x]$). Then,

$$\mathbf{E}_{[31],bdd}(x) = \mathbb{R}\mathbf{e}_3(x) \oplus \mathbb{R}\mathbf{e}_4(x),$$

Depending on the boundedness behavior of $u_{12}(x,y)$ as y varies over $\mathcal{F}_{12}[x]$ we would have either

$$\mathbf{E}_{12,bdd}(x) = \{0\} \qquad \text{or} \qquad \mathbf{E}_{12,bdd}(x) = \mathbb{R}\mathbf{e}_2(x).$$

Since $[11]' = \{11\}$ and $[12]' = \{12\}$, we have $\mathbf{E}_{[11],bdd}(x) = \mathbf{E}_{11,bdd}(x)$ and $\mathbf{E}_{[12],bdd}(x) = \mathbf{E}_{12,bdd}(x)$.

10.1*. **Bounded subspaces and synchronized exponents.** For $x \in \tilde{X}$, $y \in \tilde{X}$, let

$$\rho(x,y) = \begin{cases} |t| & \text{if } y = g_t x, \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 10.4. *For every $\eta > 0$ and $\eta' > 0$ there exists $h = h(\eta, \eta')$ such that the following holds: Suppose $\mathbf{v} \in \mathbf{E}_{ij}(x)$ and*

$$d\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}, \mathbf{E}_{i,j-1}(x)\right) > \eta'.$$

Then if $y \in \mathcal{F}_{\mathbf{v}}[x]$ and

$$\rho(y, \mathcal{F}_{ij}[x]) > h$$

then

$$d(R(x,y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)) \le \eta \|\mathbf{v}\|.$$

Proof. There exists $t \in \mathbb{R}$ such that $y' = g_t y \in \mathcal{F}_{ij}[x]$. Then

$$\rho(y, \mathcal{F}_{ij}[x]) = \rho(y, y') = |t| > h.$$

We have the orthogonal decomposition $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{w}$, where $\hat{\mathbf{v}} \in \mathbf{E}_{ij}(x)$ and $\mathbf{w} \in \mathbf{E}_{i,j-1}(x)$. Then by (9.6) we have the orthogonal decomposition.

$$R(x,y')\hat{\mathbf{v}} = e^{\lambda_{ij}(x,y')}\mathbf{v}' + \mathbf{w}', \qquad \text{where } \mathbf{v}' \in \mathbf{E}'_{ij}(y'),\ \mathbf{w}' \in \mathbf{E}_{i,j-1}(y').$$

Since $R(x,y')\mathbf{w} \in \mathbf{E}_{i,j-1}(y')$, we have

$$\|R(x,y')\mathbf{v}\|^2 = e^{2\lambda_{ij}(x,y')}\|\hat{\mathbf{v}}\|^2 + \|\mathbf{w}' + R(x,y')\mathbf{w}\|^2 \ge e^{2\lambda_{ij}(x,y')}\|\hat{\mathbf{v}}\|^2.$$

By (9.9), we have $\lambda_{ij}(x,y') = 0$. Hence,

$$\|R(x,y')\mathbf{v}\| \ge \|\hat{\mathbf{v}}\| \ge \eta'\|\mathbf{v}\|.$$

Since $y \in \mathcal{F}_{\mathbf{v}}[x]$, $\|R(x,y)\mathbf{v}\| = \|\mathbf{v}\|$. Since $|t| > h$, we have either $t > h$ or $t < -h$. If $t < -h$, then by (9.4),

$$\|\mathbf{v}\| = \|R(x,y)\mathbf{v}\| = \|(g_{-t})_* R(x,y')\mathbf{v}\| \ge e^{\kappa^{-1}h}\|R(x,y')\mathbf{v}\| \ge e^{\kappa^{-1}h}\eta'\|\mathbf{v}\|,$$

which is a contradiction if $h > \kappa \log(1/\eta')$. Hence we may assume that $t > h$. We have,

$$R(x,y)\mathbf{v} = e^{\lambda_{ij}(x,y)}\mathbf{v}'' + \mathbf{w}''$$

where $\mathbf{v}'' \in \mathbf{E}'_{ij}(y)$ with $\|\mathbf{v}''\| = \|\hat{\mathbf{v}}\|$, and $\mathbf{w}'' \in \mathbf{E}_{i,j-1}(y)$. Hence,

$$d(R(x,y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)) = e^{\lambda_{ij}(x,y)}\|\hat{\mathbf{v}}\| \le e^{\lambda_{ij}(x,y)}\|\mathbf{v}\|.$$

But,

$$\lambda_{ij}(x,y) = \lambda_{ij}(x,y') + \lambda_{ij}(y',-t) \le -\kappa^{-1} t$$

by (9.9) and Proposition 4.11. Therefore,

$$d(R(x,y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)) \le e^{-\kappa^{-1}t}\|\mathbf{v}\| \le e^{-\kappa^{-1}h}\|\mathbf{v}\|.$$

□

The bounded subspace. Fix $\theta > 0$. (We will eventually choose θ sufficiently small depending only on the dimension).

Definition 10.5. Suppose $x \in X$. A vector $\mathbf{v} \in \mathbf{E}_{ij}(x)$ is called (θ, ij)-*bounded* if there exists $C < \infty$ such that for all $\ell > 0$ and for $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x,\ell]$,

$$\|R(x,y)\mathbf{v}\| < C\|\mathbf{v}\|. \tag{10.1}$$

Remark. From the definition and (9.6), it is clear that every vector in $\mathbf{E}_{i1}(x)$ is $(i1,\theta)$-bounded for every θ. Indeed, we have $\mathbf{E}'_{i1} = \mathbf{E}_{i1}$, and $\lambda_{i1}(x,y) = 0$ for $y \in \mathcal{F}_{i1}[x]$, thus for $y \in \mathcal{F}_{i1}[x]$ and $\mathbf{v} \in E_{i1}(x)$, $\|R(x,y)\mathbf{v}\| = \|\mathbf{v}\|$.

Lemma 10.6. *The linear span of the θ/n-bounded vectors in $\mathbf{E}_{ij}(x)$ is a subspace $\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$. Any vector in this subspace is θ-bounded. Also,*

(a) *$\mathbf{E}_{ij,bdd}(x)$ is g_t-equivariant, i.e. $(g_t)_*\mathbf{E}_{ij,bdd}(x) = \mathbf{E}_{ij,bdd}(g_t x)$.*

(b) *For almost all $u \in \mathcal{B}(x)$, $\mathbf{E}_{ij,bdd}(ux) = (u)_*\mathbf{E}_{ij,bdd}(x)$.*

Proof. Let $\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$ denote the linear span of all $(\theta/n, ij)$-bounded vectors. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are any n $(\theta/n, ij)$-bounded vectors, then there exists $C > 1$ such that for $1-\theta$ fraction of y in $\mathcal{F}_{ij}[x,L]$, (10.1) holds. But then (10.1) holds (with a different C) for any linear combination of the $\mathbf{v}_i$. This shows that any vector in $\mathbf{E}_{ij,bdd}(x)$ is (θ, ij)-bounded. To show that (a) holds, suppose that $\mathbf{v} \in \mathbf{E}_{ij}(x)$ is $(\theta/n, ij)$-bounded, and $t \in \mathbb{R}$. In view of Lemma 8.2, it is enough to show that $\mathbf{v}' \equiv (g_t^{ij})_*\mathbf{v} \in \mathbf{E}_{ij}(g_t^{ij}x)$ is $(\theta/n, ij)$-bounded. Let $x' = g_t^{ij}x$.

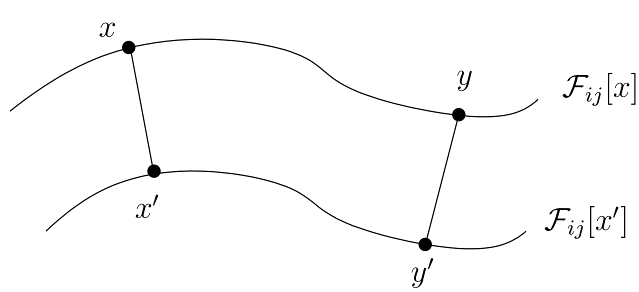

Figure 3. Proof of Lemma 10.6 (a).

By (9.4), there exists $C_1 = C_1(t)$ such that for all $z \in X$ and all $\mathbf{w} \in \mathbf{E}(z)$,

$$C_1^{-1}\|\mathbf{w}\| \le \|(g_t^{ij})_*\mathbf{w}\| \le C_1\|\mathbf{w}\|. \tag{10.2}$$

Suppose $y \in \mathcal{F}_{ij}[x, L]$ satisfies (10.1). Let $y' = g_t^{ij} y$. Then $y' \in \mathcal{F}_{ij}[x']$. Let $\mathbf{v}' = (g_t^{ij})_* \mathbf{v}$. (See Figure 3). Note that

$$R(x', y')\mathbf{v}' = R(y, y')R(x, y)R(x', x)\mathbf{v}' = R(y, y')R(x, y)\mathbf{v}$$

hence by (10.2), (10.1), and again (10.2),

$$\|R(x', y')\mathbf{v}'\| \le C_1 \|R(x, y)\mathbf{v}\| \le C_1 C \|\mathbf{v}\| \le C_1^2 C \|\mathbf{v}'\|.$$

Hence, for $y \in \mathcal{F}_{ij}[x, L]$ satisfying (10.1), $y' = g_t^{ij} y \in \mathcal{F}_{ij}[x']$ satisfies

$$\|R(x', y')\mathbf{v}'\| < C C_1^2 \|\mathbf{v}'\|. \tag{10.3}$$

Therefore, since $\mathcal{F}_{ij}[g_t^{ij} x, L + t] = g_t^{ij} \mathcal{F}_{ij}[x, L]$, we have that for $1 - \theta/n$ fraction of $y' \in \mathcal{F}_{ij}[x', L + t]$, (10.3) holds. Therefore, $\mathbf{v}'$ is $(\theta/n, ij)$-bounded. Thus, $\mathbf{E}_{ij,bdd}(x)$ is g_t-equivariant. This completes the proof of (a). Then (b) follows immediately from (a) and Lemma 9.3. $\square$

Remark 10.7. Formally, from its definition, the subspace $\mathbf{E}_{ij,bdd}(x)$ depends on the choice of θ. It is clear that as we decrease θ, the subspace $\mathbf{E}_{ij,bdd}(x)$ increases. In view of Lemma 10.6, there exists $\theta_0 > 0$ and $m \ge 0$ such that for all $\theta < \theta_0$ and almost all $x \in X$, the dimension of $\mathbf{E}_{ij,bdd}(x)$ is m. We will always choose $\theta \ll \theta_0$.

Synchronized Exponents.

Definition 10.8. Suppose $\theta > 0$. We say that $ij \in \Lambda''$ and $kr \in \Lambda''$ are *θ-synchronized* if the exists $E \subset X$ with $\nu(E) > 0$, and $C < \infty$, such that for all $x \in \pi^{-1}(E)$, for all $\ell > 0$, for at least $(1 - \theta)$-fraction of $y \in \mathcal{F}_{ij}[x, \ell]$, we have

$$\rho(y, \mathcal{F}_{kr}[x]) < C.$$

Remark 10.9. By the same argument as in the proof of Lemma 10.6 (a), if ij and kr are θ-synchronized then we can take the set E in Definition 10.8 to have full measure.

Remark 10.10. Clearly if ij and kr are θ-synchronized, then they are also θ'-synchronized for any $\theta' < \theta$. Therefore there exists $\theta_0' > 0$ such that if any pairs ij and kr are θ-synchronized for some $\theta > 0$ then they are also θ_0'-synchronized. We will always consider $\theta \ll \theta_0'$, and will sometimes use the term "synchronized" with no modifier to mean θ-synchronized for $\theta \ll \theta_0'$. Then in view of Remark 10.7 and Remark 10.9, synchronization is an equivalence relation.

If $\mathbf{v} \in \mathbf{E}(x)$, we can write

$$\mathbf{v} = \sum_{ij \in I} \mathbf{v}_{ij}, \qquad \text{where } \mathbf{v}_{ij} \in \mathbf{E}_{ij}(x), \text{ but } \mathbf{v}_{ij} \notin \mathbf{E}_{i,j-1}(x). \tag{10.4}$$

In the sum, I is a finite set of pairs ij where $i \in \tilde{\Lambda}$ and $1 \le j \le n_i$. Since for a fixed i the $\mathbf{E}_{ij}(x)$ form a flag, without loss of generality we may (and always will) assume that I contains at most one pair ij for each $i \in \tilde{\Lambda}$.

For $\mathbf{v} \in \mathbf{E}(x)$, and $y \in \mathcal{F}_{\mathbf{v}}[x]$, let

$$H(x, y) = \sup_{ij \in I} \rho(y, \mathcal{F}_{ij}[x]). \tag{10.5}$$

Lemma 10.11. *Suppose there exists a set $E \subset X$ with $\nu(E) > 0$, and $C < \infty$ such that for any $x \in E$ there exists $\mathbf{v} \in \mathbf{E}(x)$ so that for each $L > 0$, for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L]$*

$$H(x, y) < C.$$

Then, if we write $\mathbf{v} = \sum_{i \in I} \mathbf{v}_{ij}$ as in (10.4), then all $\{ij\}_{ij \in I}$ are synchronized, and also for all $ij \in I$, $\mathbf{v}_{ij} \in \mathbf{E}_{ij,bdd}(x)$.

Proof. Suppose $ij \in I$ and $kr \in I$. We have for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L]$,

$$\rho(y, \mathcal{F}_{ij}[x]) < C, \qquad \rho(y, \mathcal{F}_{kr}[x]) < C.$$

Let $y_{ij} \in \mathcal{F}_{ij}[x]$ be such that $\rho(y, \mathcal{F}_{ij}[x]) = \rho(y, y_{ij})$. Similarly, let $y_{kr} \in \mathcal{F}_{kr}[x]$ be such that $\rho(y, \mathcal{F}_{kr}[x]) = \rho(y, y_{kr})$. We have

$$\rho(y_{ij}, y_{kr}) \leq \rho(y_{ij}, y) + \rho(y, y_{kr}) \leq 2C.$$

Note that $\tilde{g}_{-L}^{\mathbf{v},x}(\mathcal{F}_{\mathbf{v}}[x, L]) = g_{-L'}^{ij}(\mathcal{F}_{ij}[x, L'])$, where L' is chosen so that $g_{-L}^{\mathbf{v}} x = g_{-L'}^{ij} x$, where the notation $\tilde{g}$ is as in (9.12). Hence, for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$, $\rho(y_{ij}, \mathcal{F}_{kr}[x]) \leq 2C$. This implies that ij and kr are synchronized.

Recall that I contains at most one j for each $i \in \tilde{\Lambda}$. Since $R(x, y)$ preserves each $\mathbf{E}_i$, and the distinct $\mathbf{E}_i$ are orthogonal, for all $y'' \in G[x]$,

$$\|R(x, y'')\mathbf{v}\|^2 = \sum_{ij \in I} \|R(x, y'')\mathbf{v}_{ij}\|^2.$$

Therefore, for each $ij \in I$, and all $y'' \in G[x]$,

$$\|R(x, y'')\mathbf{v}_{ij}\| \leq \|R(x, y'')\mathbf{v}\|.$$

In particular,

$$\|R(x, y_{ij})\mathbf{v}_{ij}\| \leq \|R(x, y_{ij})\mathbf{v}\|.$$

We have for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$, $\rho(y_{ij}, y) < C$, where $y \in \mathcal{F}_{\mathbf{v}}(x)$. We have $\|R(x, y)\mathbf{v}\| = \|\mathbf{v}\|$, and hence, by (9.4), for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x]$,

$$\|R(x, y_{ij})\mathbf{v}\| \leq C_2 \|\mathbf{v}\|.$$

Hence, $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$,

$$\|R(x, y_{ij})\mathbf{v}_{ij}\| \leq C_2 \|\mathbf{v}\|.$$

This implies that $\mathbf{v}_{ij} \in \mathbf{E}_{ij,bdd}(x)$. $\square$

We write $ij \sim kr$ if ij and kr are synchronized. With our choice of $\theta > 0$, synchronization is an equivalence relation, see Remark 10.10. We write $[ij] = \{kr : kr \sim ij\}$. Let

$$\mathbf{E}_{[ij],bdd}(x) = \bigoplus_{kr \in [ij]} \mathbf{E}_{kr,bdd}(x).$$

For $\mathbf{v} \in \mathbf{E}(x)$, write $\mathbf{v} = \sum_{ij \in I} \mathbf{v}_{ij}$, as in (10.4). Define

$$\text{height}(\mathbf{v}) = \sum_{ij \in I} (\dim \mathbf{E}) i + j$$

The height is defined so it would have the following properties:

- If $\mathbf{v} \in \mathbf{E}_{ij}(x) \setminus \mathbf{E}_{i,j-1}(x)$ and $\mathbf{w} \in \mathbf{E}_{i,j-1}(x)$ then $\text{height}(\mathbf{w}) < \text{height}(\mathbf{v})$.
- If $\mathbf{v} = \sum_{i \in I} \mathbf{v}_i$, $\mathbf{v}_i \in \mathbf{E}_i$, $\mathbf{v}_i \neq 0$, and $\mathbf{w} = \sum_{j \in J} \mathbf{w}_j$, $\mathbf{w}_j \in \mathbf{E}_j$, $\mathbf{w}_j \neq 0$, and also the cardinality of J is smaller then the cardinality of I, then $\text{height}(\mathbf{w}) < \text{height}(\mathbf{v})$.

Let $\mathcal{P}_k(x) \subset \mathbf{E}(x)$ denote the set of vectors of height at most k. This is a closed subset of $\mathbf{E}(x)$.

Lemma 10.12. *For every $\delta > 0$ every $\eta > 0$ and every $L_0 > 0$, there exists a subset $K \subset X$ of measure at least $1 - \delta$ and $L'' > 0$ such that for any $x \in K$ and any unit vector $\mathbf{v} \in \mathcal{P}_k(x)$ with $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$ and $d(\mathbf{v}, \mathcal{P}_{k-1}(x)) > \eta$, there exists $L_0 < L' < L''$ so that for least θ-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$,*

$$d\left(\frac{R(x,y)\mathbf{v}}{\|R(x,y)\mathbf{v}\|}, \mathcal{P}_{k-1}(y)\right) < \eta. \tag{10.6}$$

Proof. Suppose $C > 1$ (we will choose C depending on η). We first claim that we can choose K with $\nu(K) > 1 - \delta$ and $L'' > 0$ so that for every $x \in g_{[-1,1]}K$ and every $\mathbf{v} \in \mathcal{P}_k(x)$ such that $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$ there exists $0 < L' < L''$ so that for θ-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$,

$$H(x,y) \geq C. \tag{10.7}$$

Indeed, let $E_L \subset \mathcal{P}_k(x)$ denote the set of unit vectors $\mathbf{v} \in \mathcal{P}_k(x)$ such that for all $0 < L' < L$, for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$, $H(x,y) \leq C$. Then, the E_L are closed sets which are decreasing as L increases, and by Lemma 10.11,

$$\bigcap_{L=1}^{\infty} E_L = \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(x).$$

Let F denote subset of the unit sphere in $\mathcal{P}_k(x)$ which is the complement of the η-neighborhood of $\bigcup_{ij} \mathbf{E}_{[ij],bdd}(x)$. Then the E_L^c are an open cover of F, and since F is compact, there exists $L = L_x$ such that $F \subset E_L^c$. Now for any $\delta > 0$ we can choose L'' so that $L'' > L_x$ for all x in a set K of measure at least $(1 - \delta)$.

Now suppose $\mathbf{v} \in F$. Since $F \subset E_{L''}^c$, $\mathbf{v} \notin E_{L''}$, hence there exists $L_0 < L' < L''$ (possibly depending on $\mathbf{v}$) such that the fraction of $y \in \mathcal{F}_{\mathbf{v}}[x', L']$ which satisfies $H(x, y) \geq C$ is greater than θ. Then, (10.7) holds.

Now suppose (10.7) holds. Write

$$\mathbf{v} = \sum_{ij \in I} \mathbf{v}_{ij}$$

as in (10.4). Let

$$\mathbf{w} = R(x, y)\mathbf{v}, \qquad \mathbf{w}_{ij} = R(x, y)\mathbf{v}_{ij}.$$

Since $y \in \mathcal{F}_{\mathbf{v}}[x]$, $\|\mathbf{w}\| = \|\mathbf{v}\| = 1$. Let $ij \in I$ be such that the supremum in the definition of $H(x, y)$ is achieved for ij. If $\|\mathbf{w}_{ij}\| < \eta$ we are done, since $\mathbf{w}' = \sum_{kr \neq ij} \mathbf{w}_{kr}$ has smaller height than $\mathbf{v}$, and $d(\mathbf{w}, \mathbf{w}') < \eta$. Hence we may assume that $1 \geq \|\mathbf{w}_{ij}\| \geq \eta$.

Since $d(\mathbf{v}, \mathcal{P}_{k-1}(x)) \geq \eta$, we have

$$d(\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(x)) \geq \eta \geq \eta \|\mathbf{v}_{ij}\|.$$

where the last inequality follows from the fact that $\|\mathbf{v}_{ij}\| \leq 1$. In particular, we have $1 \geq \|\mathbf{v}_{ij}\| \geq \eta$.

Let $y' = g_t y$ be such that $y' \in \mathcal{F}_{\mathbf{v}_{ij}}[x]$. Then, in view of (9.4), $|t| \leq C_0(\eta)$, and hence $\|R(y', y)\| \leq C_0'(\eta)$.

Let $C_1 = h(\eta, \eta/C_0'(\eta))$, where $h(\cdot, \cdot)$ is as in Lemma 10.4. If $H(x, y) > C_1$ then by Lemma 10.4 applied to $\mathbf{v}_{ij}$ and $y' \in \mathcal{F}_{\mathbf{v}_{ij}}[x]$,

$$d(R(x, y')\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(y')) \leq (\eta/C_0'(\eta))\|\mathbf{v}_{ij}\| \leq \eta/C_0'(\eta).$$

Then, since $\mathbf{w}_{ij} = R(y', y)R(x, y')\mathbf{v}_{ij}$,

$$\|d(\mathbf{w}_{ij}, \mathbf{E}_{i,j-1}(y))\| \leq \|R(y', y)\| d(R(x, y')\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(y')) \leq \|R(y', y)\|(\eta/C_0'(\eta)) \leq \eta.$$

Let $\mathbf{w}_{ij}'$ be the closest vector to $\mathbf{w}_{ij}$ in $\mathbf{E}_{i,j-1}(y)$, and let $\mathbf{w}' = \mathbf{w}_{ij}' + \sum_{kr \neq ij} \mathbf{w}_{ij}$. Then $d(\mathbf{w}, \mathbf{w}') < \eta$ and $\mathbf{w}' \in \mathcal{P}_{k-1}$. □

Proof of Proposition 10.1. Recall that n is the maximal possible height of a vector. Let $\delta' = \delta/n$. Let $\eta_n = \eta$. Let $L_{n-1} = L_{n-1}(\delta', \eta_n)$ and $K_{n-1} = K_{n-1}(\delta', \eta_n)$ be chosen so that Lemma 10.12 holds for $k = n-1$, $K = K_{n-1}$, $L'' = L_{n-1}$ and $\eta = \eta_n$. Let η_{n-1} be chosen so that $\exp(NL_{n-1})\eta_{n-1} \leq \eta_n$, where N is as in Lemma 7.1. We repeat this process until we choose L_1, η_1. Let $L_0 = L_1$. Let $K = K_0 \cap \dots \cap K_{n-1}$. Then $\nu(K) > 1 - \delta$.

Let

$$E_k' = \left\{ y \in \mathcal{F}_{\mathbf{v}}[x, L] \;\;:\;\; d\left(\frac{R(x, y)\mathbf{v}}{\|R(x, y)\mathbf{v}\|}, \mathcal{P}_k(y) \cup \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y) \right) < \eta_k \right\}.$$

and let

$$E_k = \tilde{g}_{-L}(E_k'),$$

so $E_k \subset \mathcal{B}[z]$, where $z = \tilde{g}_{-L}x$. Since $E_n = \mathcal{F}_{\mathbf{v}}[x, L]$, we have $E'_n = \mathcal{B}[z]$. Let $Q = \tilde{g}_{-L}(g_{[-1,1]}K \cap \mathcal{F}_v[x, L])$. Then, by assumption,

$$|Q| \geq 1 - (\theta/2)^{n+1}. \tag{10.8}$$

By Lemma 10.12, for every point in $uz \in (E_k \cap Q) \setminus E_{k-1}$ there exists a "ball" $\mathcal{B}_t[uz]$ (where $t = L - L'$ and L' is as in Lemma 10.12) such that

$$|E_{k-1} \cap \mathcal{B}_t[uz]| \geq \theta |\mathcal{B}_t[uz]|. \tag{10.9}$$

(When we are applying Lemma 10.12 we do not have $\mathbf{v} \in \mathcal{P}_k$ but rather $d(\mathbf{v}/\|\mathbf{v}\|, \mathcal{P}_k) < \eta_k$; however by the choice of the η's and the L's this does not matter). The collection of balls $\{\mathcal{B}_t[uz]\}_{uz \in (E_k \cap Q) \setminus E_{k-1}}$ as in (10.9) are a cover of $(E_k \cap Q) \setminus E_{k-1}$. These balls satisfy the condition of Lemma 3.9 (b); hence we may choose a pairwise disjoint subcollection which still covers $(E_k \cap Q) \setminus E_{k-1}$. We get

$$|E_{k-1}| \geq \theta |E_k \cap Q| \geq (\theta/2)|E_k|,$$

where we used (10.8). Hence, $|E_0| \geq (\theta/2)^n |\mathcal{B}[z]|$. Therefore $|E'_0| \geq (\theta/2)^n |\mathcal{F}_{\mathbf{v}}[x, L]|$. Since $\mathcal{P}_0 = \emptyset$, the Proposition follows from the definition of E'_0. $\square$

10.2*. **Invariant measures on $X \times \mathbb{P}^1$.** In this subsection we prove Proposition 10.2. Recall that any bundle is measurably trivial.

Lemma 10.13. *Suppose $\mathbf{L}(x)$ is an invariant subbundle or quotient bundle of $\mathbf{H}(x)$. (In fact the arguments in this subsection apply to arbitrary vector bundles). Let $\tilde{\mu}_\ell$ be the measure on $X \times \mathbb{P}^1(\mathbf{L})$ defined by*

$$\tilde{\mu}_\ell(f) = \int_X \int_{\mathbb{P}^1(\mathbf{L})} \frac{1}{|\mathcal{F}_{ij}[x, \ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} f(x, R(y, x)\mathbf{v})\, dy\, d\rho_0(\mathbf{v})\, d\nu(x) \tag{10.10}$$

where ρ_0 is the round measure on $\mathbb{P}^1(\mathbf{L})$. Let $\hat{\mu}_\ell$ be the measure on $X \times \mathbb{P}^1(\mathbf{L})$ defined by

$$\hat{\mu}_\ell(f) = \int_X \int_{\mathbb{P}^1(\mathbf{L})} \frac{1}{|\mathcal{F}_{ij}[x, \ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} f(y, R(x, y)\mathbf{v})\, dy\, d\rho_0(\mathbf{v})\, d\nu(x). \tag{10.11}$$

Then $\hat{\mu}_\ell$ is in the same measure class as $\tilde{\mu}_\ell$, and

$$\kappa^{-2} \leq \frac{d\hat{\mu}_\ell}{d\tilde{\mu}_\ell} \leq \kappa^2, \tag{10.12}$$

where κ is as in Proposition 4.11.

Proof. Let

$$F(x, y) = \int_{\mathbb{P}^1(\mathbf{L})} f(x, R(y, x)\mathbf{v})\, d\rho_0(\mathbf{v}).$$

Then,

$$\tilde{\mu}_\ell(f) = \int_X \frac{1}{|\mathcal{F}_{ij}[x, \ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} F(x, y)\, dy\, d\nu(x) \tag{10.13}$$

$$\hat{\mu}_\ell(f) = \int_X \frac{1}{|\mathcal{F}_{ij}[x,\ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} F(y,x)\, dy\, d\nu(x) \tag{10.14}$$

Let $x' = g^{ij}_{-\ell} x$. Then, in view of Proposition 4.11, $\kappa^{-1}\, d\nu(x) \le d\nu(x') \le \kappa\, d\nu(x)$. Then,

$$\frac{1}{\kappa}\tilde{\mu}_\ell(f) = \int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F(g_\ell x', g_\ell z)\, dz\, d\nu(x') \le \kappa \tilde{\mu}_\ell(f),$$

and

$$\frac{1}{\kappa}\hat{\mu}_\ell(f) = \int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F(g_\ell z, g_\ell x')\, dz\, d\nu(x') \le \kappa \hat{\mu}_\ell(f)$$

Let X'' consist of one point from each $\mathcal{B}[x]$. We now disintegrate $d\nu(x') = d\beta(x'')dz'$ where $x'' \in X''$, $z' \in \mathcal{B}[x']$. Then,

$$\begin{aligned}\int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F(g_\ell x', g_\ell z)\, dz\, d\nu(x') &= \int_{X''} \int_{\mathcal{B}[x'']\times\mathcal{B}[x'']} F(g_\ell z', g_\ell z)\, dz'\, dz\, d\beta(x'') \\ &= \int_{X''} \int_{\mathcal{B}[x'']\times\mathcal{B}[x'']} F(g_\ell z, g_\ell z')\, dz'\, dz\, d\beta(x'') \\ &= \int_X \frac{1}{|\mathcal{B}[x']|} \int_{\mathcal{B}[x']} F(g_\ell z, g_\ell x')\, dz\, d\nu(x').\end{aligned}$$

Now (10.12) follows from (10.13) and (10.14). □

Lemma 10.14. *Let $\tilde{\mu}_\infty$ be any weak-star limit of the measures $\tilde{\mu}_\ell$. Then,*

(a) *We may disintegrate $d\tilde{\mu}_\infty(x,\mathbf{v}) = d\nu(x)\, d\lambda_x(\mathbf{v})$, where for each $x \in X$, λ_x is a measure on $\mathbb{P}^1(\mathbf{L})$.*

(b) *For $x \in \tilde{X}$ and $y \in \mathcal{F}_{ij}[x]$,*

$$\lambda_y = R(x,y)_* \lambda_x,$$

(where to simplify notation, we write λ_x and λ_y instead of $\lambda_{\pi(x)}$ and $\lambda_{\pi(y)}$).

(c) *Let $\mathbf{F} \subset \mathbb{P}^1(\mathbf{L})$ be a finite union of subspaces. For $\eta > 0$ let*

$$\mathbf{F}_\eta = \{\mathbf{v} \in \mathbb{P}^1(\mathbf{L}) \;:\; d(\mathbf{v},\mathbf{F}) \ge \eta\}.$$

Then, for any $t \in \mathbb{R}$ there exists $c = c(t,\mathbf{F}) > 0$ such that

$$\lambda_{g_t^{ij}x}((g_t^{ij}\mathbf{F})_{c\eta}) \ge c\lambda_x(\mathbf{F}_\eta).$$

Consequently, λ_x is supported on $\mathbf{F}$ if and only if $\lambda_{g_t^{ij}x}$ is supported on $g_t^{ij}\mathbf{F}$.

(d) *For almost all $x \in X$ there exist a measure ψ_x on $\mathbb{P}^1(\mathbf{L})$ such that*

$$\lambda_x = h(x)\psi_x$$

for some $h(x) \in \mathrm{SL}(\mathbf{L})$, and also for almost all $y \in \mathcal{F}_{ij}[x]$, $\psi_y = \psi_x$ (so that ψ is constant on the leaves $\mathcal{F}_{ij}$).

Proof. If $f(x, \mathbf{v})$ is independent of the second variable, then it is clear from the definition of $\tilde{\mu}_\ell$ that $\tilde{\mu}_\ell(f) = \int_X f\, d\nu$. This implies (a). To prove (b), note that $R(y', y) = R(x, y)R(y', x)$. Then,

$$\begin{aligned}\lambda_y &= \lim_{k\to\infty} \frac{1}{|\mathcal{F}_{ij}[y, \ell_k]|} \int_{\mathcal{F}_{ij}[y,\ell_k]} (R(y', y)_* \rho_0)\; dy' \\ &= R(x, y)_* \lim_{k\to\infty} \frac{1}{|\mathcal{F}_{ij}[y, \ell_k]|} \int_{\mathcal{F}_{ij}[y,\ell_k]} (R(y', x)_* \rho_0)\; dy' \\ &= R(x, y)_* \lim_{k\to\infty} \frac{1}{|\mathcal{F}_{ij}[x, \ell_k]|} \int_{\mathcal{F}_{ij}[x,\ell_k]} (R(y', x)_* \rho_0)\; dy' \\ &= R(x, y)_* \lambda_x\end{aligned}$$

where to pass from the second line to the third we used the fact that $\mathcal{F}_{ij}[x, \ell] = \mathcal{F}_{ij}[y, \ell]$ for ℓ large enough. This completes the proof of (b).

To prove (c), let $x' = g_t^{ij} x$, $y' = g_t^{ij} y$. We have

$$R(y', x') = R(x, x')R(y, x)R(y', y).$$

Since $\|R(x, x')^{-1}\| \le c^{-1}$, where c depends on t, we have $R(x, x')^{-1}\mathbf{F}_{c\eta} \subset \mathbf{F}_\eta$. Then,

$$\begin{aligned}\rho_0\{\mathbf{v} \;:\; R(y', x')\mathbf{v} \in \mathbf{F}_{c\eta}\} &= \rho_0\{\mathbf{v} \;:\; R(y, x)R(y', y) \in R(x, x')^{-1}\mathbf{F}_{c\eta}\} \\ &\ge \rho_0\{\mathbf{v} \;:\; R(y, x)R(y', y)\mathbf{v} \in \mathbf{F}_\eta\} \\ &= \rho_0\{R(y, y')^{-1}\mathbf{w} \;:\; R(y, x)\mathbf{w} \in \mathbf{F}_\eta\} \\ &= R(y, y^{-1})_* \rho_0\{\mathbf{w} \;:\; R(y, x)\mathbf{w} \in \mathbf{F}_\eta\} \\ &\ge c_2 \rho_0\{\mathbf{w} \;:\; R(y, x)\mathbf{w} \in \mathbf{F}_\eta\}.\end{aligned}$$

Substituting into (10.10) completes the proof of (c).

To prove part (d), let $\mathcal{M}$ denote the space of measures on $\mathbb{P}^1(\mathbf{L})$. Recall that by [Zi2, Theorem 3.2.6] the orbits of the special linear group $\mathrm{SL}(\mathbf{L})$ on $\mathcal{M}$ are locally closed. Then, by [Ef, Theorem 2.9 (13), Theorem 2.6(5)] [1] there exists a Borel cross section $\phi : \mathcal{M}/\operatorname{SL}(\mathbf{L}) \to \mathcal{M}$. Then, let $\psi_x = \phi(\pi(\lambda_x))$ where $\pi : \mathcal{M} \to \mathcal{M}/\operatorname{SL}(\mathbf{L})$ is the quotient map. □

We also recall the following well known Lemma of Furstenberg (see e.g. [Zi2, Lemma 3.2.1]):

Lemma 10.15. *Let* $\mathbf{L}$ *be a vector space, and suppose* μ *and* ν *are two probability measures on* $\mathbb{P}^1(\mathbf{L})$. *Suppose* $g_i \in \mathrm{SL}(\mathbf{L})$ *are such that* $g_i \to \infty$ *and* $g_i\mu \to \nu$. *Then the support of* ν *is contained a union of two proper subspaces of* $\mathbf{L}$.

In particular, if the support of a measure ν *on* $\mathbb{P}^1(\mathbf{L})$ *is not contained in a union of two proper subspaces, then the stabilizer of* ν *in* $\mathrm{SL}(\mathbf{L})$ *is bounded.*

[1] The "condition C" of [Ef] is satisfied since $\mathrm{SL}(\mathbf{L})$ is locally compact and $\mathcal{M}$ is Hausdorff.

Lemma 10.16. *Suppose* $\mathbf{L}$ *is either a subbundle or a quotient bundle of* $\mathbf{H}$. *Suppose that* $\theta > 0$, *and suppose that for all* $\delta > 0$ *there exists a set* $K \subset X$ *with* $\nu(K) > 1-\delta$ *and a constant* $C_1 < \infty$, *such that such that for all* $\ell > 0$ *and at least* $(1-\theta)$*-fraction of* $y \in \mathcal{F}_{ij}(x,\ell)$,

$$\|R(x,y)\mathbf{v}\| \le C_1\|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in \mathbf{L}. \tag{10.15}$$

Then for all $\ell > 0$ *there exists a subset* $K''(\ell) \subset X$ *with* $\nu(K''(\ell)) > 1 - c(\delta)$ *where* $c(\delta) \to 0$ *as* $\delta \to 0$, *and there exists* $\theta'' = \theta''(\theta)$ *with* $\theta'' \to 0$ *as* $\theta \to 0$ *such that for all* $x \in K''(\ell)$, *for at least* $(1-\theta'')$*-fraction of* $y \in \mathcal{F}_{ij}(x,\ell)$,

$$C_1^{-1}\|\mathbf{v}\| \le \|R(x,y)\mathbf{v}\| \le C_1\|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in \mathbf{L}. \tag{10.16}$$

Proof. By Lemma 10.13 (with f the characteristic function of $K \times \mathbb{P}^1(\mathbf{L})$), we have $\hat{\mu}_\ell(K) \ge (1-\kappa^2\delta)$. Therefore there exists a subset $K'(\ell) \subset X$ with $\nu(K'(\ell)) \ge 1-(\kappa^2\delta)^{1/2}$ such that such that for all $x \in K'(\ell)$,

$$|\mathcal{F}_{ij}[x,\ell] \cap K| \ge (1-(\kappa^2\delta)^{1/2})|\mathcal{F}_{ij}[x,\ell]|.$$

For $x_0 \in X$, let

$$Z_\ell[x_0] = \{(x,y) \in \mathcal{F}_{ij}[x,\ell] \times \mathcal{F}_{ij}[x,\ell] \;:\; x \in K, \quad y \in K, \quad \text{and (10.15) holds }\}.$$

Then, if $x_0 \in K'(\ell)$ and $\theta' = \theta + (\kappa^2\delta)^{1/2}$ then, by Fubini's theorem,

$$|Z_\ell[x_0]| \ge (1-\theta')|\mathcal{F}_{ij}[x,\ell] \times \mathcal{F}_{ij}[x,\ell]|.$$

Let

$$Z_\ell[x_0]^t = \{(x,y) \in \mathcal{F}_{ij}[x,\ell] \times \mathcal{F}_{ij}[x,\ell] \;:\; (y,x) \in Z_\ell[x_0]\}.$$

Then, for $x_0 \in K'(\ell)$,

$$|Z_\ell[x_0] \cap Z_\ell[x_0]^t| \ge (1-2\theta')|\mathcal{F}_{ij}[x,\ell] \times \mathcal{F}_{ij}[x,\ell]|.$$

For $x \in \mathcal{F}_{ij}[x_0,\ell]$, let

$$Y'_\ell(x) = \{y \in \mathcal{F}_{ij}[x,\ell] \;:\; (x,y) \in Z_\ell[x] \cap Z_\ell[x]^t\}.$$

Therefore, by Fubini's theorem, for all $x_0 \in K'(\ell)$ and $\theta'' = (2\theta')^{1/2}$,

$$|\{x \in \mathcal{F}_{ij}[x_0,\ell] \;:\; |Y'_\ell(x)| \ge (1-\theta'')|\mathcal{F}_{ij}[x_0,\ell]|\}| \ge (1-\theta'')|\mathcal{F}_{ij}[x_0,\ell]|. \tag{10.17}$$

(Note that $\mathcal{F}_{ij}[x_0,\ell] = \mathcal{F}_{ij}[x,\ell]$.) Let

$$K''(\ell) = \{x \in X \;:\; |Y'_\ell(x)| \ge (1-\theta'')|\mathcal{F}_{ij}[x,\ell]|\}.$$

Therefore, by (10.17), for all $x_0 \in K'(\ell)$,

$$|\mathcal{F}_{ij}[x_0,\ell] \cap K''(\ell)| \ge (1-\theta'')|\mathcal{F}_{ij}[x_0,\ell]|.$$

Then, by the definition of $\hat{\mu}_\ell$,

$$\hat{\mu}_\ell(K''(\ell) \times \mathbb{P}^1(\mathbf{L})) \ge (1-\theta'')\nu(K'(\ell)) \ge (1-2\theta''),$$

and therefore, by Lemma 10.13,

$$\nu(K''(\ell)) = \tilde{\mu}_\ell(K''(\ell) \times \mathbb{P}^1(\mathbf{L})) \ge \hat{\mu}_\ell(K''(\ell) \times \mathbb{P}^1(\mathbf{L})) \ge (1-2\kappa^2\theta'').$$

Now, for $x \in K''(\ell)$, and $y \in Y'_\ell(x)$, (10.16) holds. $\square$

Lemma 10.17. *Suppose* $\mathbf{L}(x) = \mathbf{E}_{ij,bdd}(x)$. *Then there exists a* Γ*-invariant function* $C : \tilde{X} \to \mathbb{R}^+$ *finite almost everywhere such that for all* $x \in \tilde{X}$, *all* $\mathbf{v} \in \mathbf{L}(x)$, *and all* $y \in \mathcal{F}_j[x]$,

$$C(x)^{-1}C(y)^{-1}\|\mathbf{v}\| \leq \|R(x,y)\mathbf{v}\| \leq C(x)C(y)\|\mathbf{v}\|,$$

Proof. Let $\tilde{\mu}_\ell$ and $\hat{\mu}_\ell$ be as in Lemma 10.13. Take a sequence $\ell_k \to \infty$ such that $\tilde{\mu}_{\ell_k} \to \tilde{\mu}_\infty$, and $\hat{\mu}_{\ell_k} \to \hat{\mu}_\infty$. Then by Lemma 10.14 (a), we have $d\tilde{\mu}_\infty(x,\mathbf{v}) = d\nu(x)\, d\lambda_x(\mathbf{v})$ where λ_x is a measure on $\mathbb{P}^1(\mathbf{L})$. We will show that for almost all $x \in X$, λ_x is not supported on a union of two subspaces.

Suppose not; then for a set of positive measure λ_x is supported on $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$, where $\mathbf{F}_1(x)$ and $\mathbf{F}_2(x)$ are subspaces of $\mathbf{L}(x)$. Then, by Lemma 10.14 (c), $\mathbf{F}_1(x)$ and $\mathbf{F}_2(x)$ are g_t-invariant, and then by Proposition 4.4 (a) (or Lemma 10.14 (b)), the $\mathbf{F}_i(x)$ are also U^+-invariant. Also, in view of the ergodicity of g_t and Lemma 10.14 (c), λ_x is supported on $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$ for almost all $x \in X$.

Fix $\delta > 0$ (which will be chosen sufficiently small later). Suppose $\ell > 0$ is arbitrary. Since $\mathbf{L} = \mathbf{E}_{ij,bdd}$, there exists a constant C_1 and a compact subset $K \subset X$ with $\nu(K) > 1 - \delta$ and for each $x \in K$ a subset $Y_\ell(x)$ of $\mathcal{F}_{ij}[x,\ell]$ with $|Y_\ell(x)| \geq (1-\theta)|\mathcal{F}_{ij}[x,\ell]|$, such that for $x \in K$ and $y \in Y_\ell(x) \cap K$ we have

$$\|R(x,y)\mathbf{v}\| \leq C_1\|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in \mathbf{L}.$$

Therefore by Lemma 10.16, there exists $\theta'' > 0$, $K''(\ell) \subset X$ and for each $x \in K''(\ell)$ a subset $Y'_\ell(x) \subset \mathcal{F}_{ij}(x,\ell)$ with $|Y'_\ell(x)| \geq (1-\theta'')|\mathcal{F}_{ij}(x,\ell)|$ such that for $x \in K''(\ell)$ and $y \in Y'_\ell(x)$, (10.16) holds.

Let

$$\mathbf{Z}(x,\eta) = \{\mathbf{v} \in \mathbb{P}^1(\mathbf{L}) \ : \ d(\mathbf{v}, \mathbf{F}_1(x) \cup \mathbf{F}_2(x)) \geq \eta\}.$$

We may choose $\eta > 0$ small enough so that for all $x \in X$,

$$\rho_0(\mathbf{Z}(x, C_1\eta)) > 1/2.$$

Let

$$S(\eta) = \{(x,\mathbf{v}) \ : \ x \in X, \quad \mathbf{v} \in \mathbf{Z}(x,\eta)\}$$

Let f denote the characteristic function of the set

$$\{(x,\mathbf{v}) \ : \ x \in K''(\ell), \quad \mathbf{v} \in \mathbf{Z}(x,\eta)\} \subset S(\eta).$$

We now claim that for any ℓ,

$$\tilde{\mu}_\ell(f) \geq (1 - 2\kappa^2\theta'')(1-\theta'')(1/2). \tag{10.18}$$

Indeed, if we restrict in (10.10) to $x \in K''(\ell)$, $y \in Y'_\ell(x)$, and $\mathbf{v} \in \mathbf{Z}(x, C_1\eta)$, then by (10.16), $f(x, R(x,y)\mathbf{v}) = 1$. This implies (10.18). Thus, (provided $\delta > 0$ and $\theta > 0$ in Definition 10.5 are sufficiently small), there exists $c_0 > 0$ such that for all ℓ, $\tilde{\mu}_\ell(S(\eta)) \geq c_0 > 0$. Therefore, $\tilde{\mu}_\infty(S(\eta)) > 0$, which is a contradiction to the fact that λ_x is supported on $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$.

Thus, for almost all x, λ_x is not supported on a union of two subspaces. Thus the same holds for the measure ψ_x of Lemma 10.14 (d). By combining (b) and (d) of Lemma 10.14 we see that for almost all x and almost all $y \in \mathcal{F}_{ij}[x]$,

$$R(x,y)h(x)\psi_x = h(y)\psi_x,$$

hence $h(y)^{-1}R(x,y)h(x)$ stabilizes ψ_x. Hence by Lemma 10.15,

$$h(y)^{-1}R(x,y)h(x) \in K(x)$$

where $K(x)$ is a compact subset of $\mathrm{SL}(\mathbf{L})$. Thus, $R(x,y) \in h(y)K(x)h(x)^{-1}$, and thus

$$\|R(x,y)\| \leq C(x)C(y).$$

Since $R(x,y)^{-1} = R(y,x)$, we get, by exchanging x and y,

$$\|R(x,y)^{-1}\| \leq C(x)C(y).$$

This implies the statement of the lemma. $\square$

Lemma 10.18. *Suppose that for all $\delta > 0$ there exists a constant $C > 0$ and a compact subset $K \subset X$ with $\nu(K) > 1-\delta$ and for each $\ell > 0$ and $x \in K$ a subset $Y_\ell(x)$ of $\mathcal{F}_{ij}[x,\ell]$ with $|Y_\ell(x)| \geq (1-\theta)|\mathcal{F}_{ij}[x,\ell]|$, such that for $x \in K$ and $y \in Y_\ell(x)$ we have*

$$\lambda_{kr}(x,y) \leq C. \tag{10.19}$$

(Recall that $\lambda_{ij}(x,y) = 0$ for all $y \in \mathcal{F}_{ij}[x]$.) Then, ij and kr are synchronized, and there exists a function $C : X \to \mathbb{R}^+$ finite ν-almost everywhere such that for all $x \in X$, and all $y \in \mathcal{F}_{ij}[x]$,

$$\rho(y, \mathcal{F}_{kr}[x]) \leq C(x)C(y). \tag{10.20}$$

Proof. The proof is a simplified version of the proof of Lemma 10.17. Let $\mathbf{L}_1 = \mathbf{E}_{ij}/\mathbf{E}_{i,j-1}$, $\mathbf{L}_2 = \mathbf{E}_{kr}/\mathbf{E}_{k,r-1}$, and $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$.

We have, for $y \in G[x]$, and $(\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{L}$,

$$R(x,y)(\bar{\mathbf{v}}, \bar{\mathbf{w}}) = (e^{\lambda_{ij}(x,y)}\bar{\mathbf{v}}', e^{\lambda_{kr}}\bar{\mathbf{w}}'), \qquad \text{where } \|\bar{\mathbf{v}}'\| = \|\bar{\mathbf{v}}\| \text{ and } \|\bar{\mathbf{v}}'\| = \|\bar{\mathbf{v}}\|. \tag{10.21}$$

Recall that $\lambda_{ij}(x,y) = 0$ for all $y \in \mathcal{F}_{ij}[x]$. Therefore, (10.19) implies that for all $x \in K$, all $\ell > 0$ and all $y \in Y_\ell(x)$,

$$\|R(x,y)(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq C_1\|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\|.$$

Therefore, by Lemma 10.16, there exists a subset $K''(\ell) \subset X$ with $\nu(K''(\ell)) > 1-c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$, and for each $x \in K''(\ell)$ a subset $Y'_\ell \subset \mathcal{F}_{ij}(x,\ell)$ such that

$$C_1^{-1}\|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq \|R(x,y)(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq C_1\|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\|.$$

This implies that for $x \in K''(m)$, $y \in Y'_m(x)$,

$$|\lambda_{kl}(x,y)| = |\lambda_{ij}(x,y) - \lambda_{kl}(x,y)| \leq C_1.$$

Therefore ij and kr are synchronized.

Let $\tilde{\mu}_\ell$ and $\hat{\mu}_\ell$ be as in Lemma 10.13. Take a sequence $\ell_m \to \infty$ such that $\tilde{\mu}_{\ell_m} \to \tilde{\mu}_\infty$, and $\hat{\mu}_{\ell_m} \to \hat{\nu}_\infty$. Then by Lemma 10.14 (a), we have $d\tilde{\mu}_\infty(x, \mathbf{v}) = d\nu(x)\, d\lambda_x(\mathbf{v})$ where λ_x is a measure on $\mathbb{P}^1(\mathbf{L})$. We will show that for almost all $x \in X$, λ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$.

Suppose that for a set of positive measure λ_x is supported on $(\mathbf{L}_1 \times \{0\}) \cup (\{0\} \times \mathbf{L}_2)$. Then, in view of the ergodicity of g_t and Lemma 10.14 (c), λ_x is supported on $(\mathbf{L}_1 \times \{0\}) \cup (\{0\} \times \mathbf{L}_2)$ for almost all $x \in X$. Let

$$\mathbf{Z}(x, \eta) = \{(\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{L}(x), \quad \|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| = 1, \quad d(\bar{\mathbf{v}}, \mathbf{L}_1) \geq \eta, \quad d(\bar{\mathbf{w}}, \mathbf{L}_2) \geq \eta\}.$$

and let

$$S(\eta) = \{(x, (\bar{\mathbf{v}}, \bar{\mathbf{w}})) \ : \ x \in X, (\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{Z}(x, \eta).$$

Then we have $\tilde{\mu}_\infty(S(\eta)) = 0$. Therefore, by Lemma 10.13, $\hat{\mu}_\infty(S(\eta)) = 0$. Therefore, by (10.21), for $x \in K$ and $y \in Y_{\ell_m}(x)$,

$$R(x, y)\, \mathbf{Z}(x, C_1\eta) \subset \mathbf{Z}(y, \eta). \tag{10.22}$$

Choose η so that $\rho_0(C_1\eta) > (1/2)$. Let f be the characteristic function of $S(\eta)$. Then, if we restrict in (10.11) to $x \in K$, $y \in Y_{\ell_m}(x)$, and $\mathbf{v} \in \mathbf{Z}(x, C_1\eta)$, then by (10.22), $f(x, R(x, y)\mathbf{v}) = 1$. This implies that for all m,

$$\hat{\mu_{\ell_m}}(S(\eta)) \geq \nu(K)(1 - \theta)(1/2).$$

Hence $\hat{\mu}_\infty(S(\eta)) > 0$ which is a contradiction. Therefore, for almost all x, λ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$. Thus the same holds for the measure ψ_x of Lemma 10.14 (d). By combining (b) and (d) of Lemma 10.14 we see that for almost all x and almost all $y \in \mathcal{F}_{ij}[x]$,

$$R(x, y)h(x)\psi_x = h(y)\psi_x,$$

hence $h(y)^{-1}R(x, y)h(x)$ stabilizes ψ_x. Hence by Lemma 10.15,

$$h(y)^{-1}R(x, y)h(x) \in K(x)$$

where $K(x)$ is a compact subset of $\mathrm{SL}(\mathbf{L})$. Thus, $R(x, y) \in h(y)K(x)h(x)^{-1}$, and thus

$$\|R(x, y)\| \leq C(x)C(y).$$

Note that get by reversing x and y we get $\|R(x, y)\|^{-1} \leq C(x)C(y)$. Therefore, by (10.21),

$$|\lambda_{ij}(x, y) - \lambda_{kr}(x, y)| \leq C(x)C(y).$$

□

Proof of Proposition 10.2. This follows immediately from Lemma 10.18 and Lemma 10.17. □

Proof of Proposition 10.3. Let K'' be as in Proposition 8.5 (b) with $\delta = \theta$. We may assume that the conull set Ψ in Proposition 10.3 is such so that for $x \in \Psi$, $g_{-t}x \in K''$ for arbitrarily large $t > 0$.

Suppose $y \in \mathcal{F}_{ij}[x,\ell]$. We may write

$$y = g_{t'}^{ij} u g_{-t'}^{ij} x = g_{s'} u g_{-t} x.$$

Then, by the multiplicative ergodic theorem, assuming t' is sufficiently large and $g_{-t}x \in K''$,

$$|s' - \lambda_i t'| \le \epsilon t \quad \text{and} \quad |t - \lambda_i t'| \le \epsilon t. \tag{10.23}$$

Now suppose $\mathbf{v} \in \mathbf{H}(x)$. Note that if $\|R(x,y)\mathbf{v}| \le C\|\mathbf{v}\|$, and s is as in Proposition 8.5, then $s > s' - O(1)$ (where the implied constant depends on C.) Therefore, in view of (10.23), (8.16) holds. Thus, by Proposition 8.5 (b), we have $\mathbf{v} \in \mathbf{E}(x)$. Thus, we can write

$$\mathbf{v} = \sum_{kr \in I} \mathbf{v}_{kr}$$

where the indexing set I contains at most one r for each $k \in \tilde{\Lambda}$. Let K be as in Lemma 10.18, and let Ψ be such that for $x \in \Psi$, $g_{-t}x \in K$ infinitely often. Note that for $y \in \mathcal{F}_{ij}[x]$,

$$\|R(x,y)\mathbf{v}\| \ge \|R(x,y)\mathbf{v}_{kr}\| \ge e^{\lambda_{kr}(x,y)}\|\mathbf{v}_{kr}\|.$$

Then, by Lemma 10.18, for all $kr \in I$, kr and ij are synchronized, i.e. $kr \in [ij]$. Now, by Proposition 10.2 (or Definition 10.5), $\mathbf{v}_{kr}(x) \in \mathbf{E}_{kr,bdd}(x)$. Therefore, $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$. $\square$

11. Equivalence relations on W^+.

Let GSpc denote the space of generalized subspaces of W^+. We have a map $\mathcal{U}_x : \mathcal{H}_+(x) \times W^+(x) \to \text{GSpc}$ taking the pair (M,v) to the generalized subspace it parametrizes. Let $\mathcal{U}_x^{-1}$ denote the inverse of this map (given a Lyapunov-adapted transversal $Z(x)$).

Let

$$\mathcal{E}_{ij}[x] = \{\mathcal{Q} \in \text{GSpc} \;:\; \mathbf{j}(\mathcal{U}_x^{-1}(\mathcal{Q})) \in \mathbf{E}_{[ij],bdd}(x).\}$$

Motivation. In view of Proposition 10.2 and Lemma 6.5 (b), the conditions that $\mathcal{Q} \in \mathcal{E}_{ij}[x]$ and $hd_x(\mathcal{Q}, U^+[x]) = O(\epsilon)$ imply the following: for "most" $y \in \mathcal{F}_{ij}[x]$,

$$hd_y(R(x,y)\mathcal{Q}, U^+[y]) = O(\epsilon).$$

In view of Proposition 10.1, we can ensure, in the notation of §2.3 that for some $ij \in \tilde{\Lambda}$, q_2' is close to $\mathcal{E}_{ij}(q_2)$; then in the limit we would have $\tilde{q}_2' \in \mathcal{E}_{ij}(\tilde{q}_2)$.

A partition of $W^+[x]$. Pick disjoint open sets of diameter at most 1, so that their union is conull in the intersection of $W^+[x]$ with the fundamental domain. We denote the set containing x by $B[x]$, and let $B(x) = \{v \in W^+(x) \;:\; v + x \in B[x]\}$.

Equivalence relations. Fix $x_0 \in X$. For x, $x' \in W^+[x_0]$ we say that

$$x' \sim_{ij} x \text{ if } x' \in B[x] \text{ and } U^+[x'] \in \mathcal{E}_{ij}[x].$$

Proposition 11.1. *The relation $\sim_{ij}$ is a (measurable) equivalence relation.*

The main part of the proof of Proposition 11.1 is the following:

Lemma 11.2. *There exists a subset $\Psi \subset X$ with $\nu(\Psi) = 1$ such that for any $ij \in \tilde{\Lambda}$, if $x_0 \in \Psi$, $x_1 \in \Psi$, $d(x_0, x_1) < 1$, and $U^+[x_1] \in \mathcal{E}_{ij}[x_0]$, then $\mathcal{E}_{ij}[x_1] = \mathcal{E}_{ij}[x_0]$.*

Warning. We will consider the condition $x' \sim_{ij} x$ to be undefined unless x and x' both belong to the set Ψ of Lemma 11.2.

Proof of Proposition 11.1, assuming Lemma 11.2. We have $0 \in \mathcal{E}_{[ij],bdd}(x)$, therefore,

$$U^+[x] \in \mathcal{E}_{ij}[x]. \tag{11.1}$$

Thus $x \sim_{ij} x$.

Suppose $x' \sim_{ij} x$. Then, $x' \in B[x]$, and so $x \in B[x']$. By (11.1), $U^+[x] \in \mathcal{E}_{ij}[x]$, and by Lemma 11.2, $\mathcal{E}_{ij}[x'] = \mathcal{E}_{ij}[x]$. Therefore, $U^+[x] \in \mathcal{E}_{ij}[x']$, and thus $x \sim_{ij} x'$.

Now suppose $x' \sim_{ij} x$ and $x'' \sim_{ij} x'$. Then, $x'' \in B[x]$. Also, $U^+[x''] \in \mathcal{E}_{ij}[x'] = \mathcal{E}_{ij}[x]$, therefore $x'' \sim_{ij} x$. □

Remark. By Lemma 11.2, for $x, x' \in \Psi$, $x' \sim_{ij} x$ if and only if $x' \in B[x]$ and $\mathcal{E}_{ij}[x'] = \mathcal{E}_{ij}[x]$.

Outline of the proof of Lemma 11.2. Intuitively, the condition $U^+[x_1] \in \mathcal{E}_{ij}[x_0]$ is the same as "$\mathcal{F}_{ij}[x_1]$ and $\mathcal{F}_{ij}[x_0]$ stay close", and "$U^+[x_1]$ and $U^+[x_0]$ stay close as we travel along $\mathcal{F}_{ij}[x_0]$ or $\mathcal{F}_{ij}[x_1]$", which is clearly an equivalence relation. We give some more detail below.

Fix $\epsilon \ll 1$. Suppose x_1 is ϵ-close to x_0, and

$$hd_{x_0}(U^+[x_1], U^+[x_0]) = \epsilon.$$

Then, by Lemma 6.5 (b),

$$\mathbf{j}(\mathcal{U}_x^{-1}(U^+[x_1])) = O(\epsilon).$$

We are given that $U^+[x_1] \in \mathcal{E}_{ij}[x_0]$, thus $\mathbf{j}(\mathcal{U}_{x_0}^{-1}(U^+[x_1])) \in \mathbf{E}_{[ij],bdd}(x_0)$. Then, by Proposition 10.2, for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$\|R(x_0, y_0)\mathbf{j}(\mathcal{U}_{x_0}^{-1}(U^+[x_1]))\| = O(\epsilon).$$

We have

$$R(x_0, y_0)\mathbf{j}(\mathcal{U}_{x_0}^{-1}(U^+[x_1])) = \mathbf{j}(\mathcal{U}_{y_0}^{-1}(U^+[y_1'])),$$

for some $y_1' \in G[x_1]$. Then, by Lemma 6.5 (b), for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$hd_{y_0}(U^+[y_1'], U^+[y_0]) = O(\epsilon) \quad \text{for some } y_1' \in G[x_1].$$

It is not difficult to show that y_1' is near a point $y_1 \in \mathcal{F}_{ij}[x_1]$. Thus, for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$hd_{y_0}(U^+[y_1], U^+[y_0]) = O(\epsilon) \quad \text{for some } y_1 \in \mathcal{F}_{ij}[x_1]. \tag{11.2}$$

Thus, most of the time $\mathcal{F}_{ij}[x_0]$ and $\mathcal{F}_{ij}[x_1]$ remain close, and also that for most $y_0 \in \mathcal{F}_{ij}[x_0]$, $U^+[y_1]$ and $U^+[y_0]$ remain close, for some $y_1 \in \mathcal{F}_{ij}[x_1]$.

Now suppose $\mathcal{Q}_1 \in \mathcal{E}_{ij}[x_1]$, and

$$hd_{x_1}(\mathcal{Q}_1, U^+[x_1]) = O(\epsilon).$$

Then, $\mathbf{j}(\mathcal{U}_{x_1}^{-1}(\mathcal{Q}_1)) \in \mathbf{E}_{[ij],bdd}(x_1)$, and thus, for most $y_1 \in \mathcal{F}_{ij}[x_1]$, using Proposition 10.2 and Lemma 6.5 (b) twice as above, we get that for most $y_1 \in \mathcal{F}_{ij}[x_1]$,

$$hd_{y_1}(R(x_1, y_1)\mathcal{Q}_1, U^+[y_1]) = O(\epsilon). \tag{11.3}$$

In our notation, $R(x_1, y_1)\mathcal{Q}_1$ is the same generalized subspace (i.e. the same subset of W^+) as $R(x_0, y_0)\mathcal{Q}_1$ for $y_0 \in \mathcal{F}_{ij}[x_0]$ close to y_1. Then, from (11.2) and (11.3), for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$hd_{y_0}(R(x_0, y_0)\mathcal{Q}_1, U^+[y_0]) = O(\epsilon).$$

Thus, using Lemma 6.5 (b) again, we get that for most $y_0 \in \mathcal{F}_{ij}[x_0]$,

$$\|R(x_0, y_0)\mathbf{j}(\mathcal{U}_{x_0}^{-1}(\mathcal{Q}_1))\| = O(\epsilon).$$

By Proposition 10.3, this implies that $\mathbf{j}(\mathcal{U}_{x_0}^{-1}(\mathcal{Q}_1)) \in \mathbf{E}_{[ij],bdd}(x_0)$, and thus $\mathcal{Q}_1 \in \mathcal{E}_{ij}[x_0]$. Thus, $\mathcal{E}_{ij}[x_1] \subset \mathcal{E}_{ij}[x_0]$.

Conversely, if $\mathcal{Q}_0 \in \mathcal{E}_{ij}[x_0]$, then the same argument shows that $\mathcal{Q}_0 \in \mathcal{E}_{ij}[x_1]$. Therefore, $\mathcal{E}_{ij}[x_0] = \mathcal{E}_{ij}[x_1]$. $\square$

The (tedious) formal verification of Lemma 11.2 is given in §11.1* below.

The equivalence classes $\mathcal{C}_{ij}[x]$. The equivalence relation $\sim_{ij}$ is clearly measurable. We define

$$\mathcal{C}_{ij}[x] = \{x' \in B[x] \ : \ x' \sim_{ij} x\}.$$

We define a ν-measurable σ-algebra $\mathcal{C}_{ij}$ so that (after possibly modifying the $\mathcal{C}_{ij}[x]$ on a set of ν-measure 0) the atoms of $\mathcal{C}_{ij}$ are the sets $\mathcal{C}_{ij}[x]$. By [EL, Proposition 5.8], we may, after possibly replacing $\mathcal{C}_{ij}$ by an ν-equivalent σ-algebra, assume that $\mathcal{C}_{ij}$ is countably generated. (See e.g. [CK] for the definitions and a discussion of related issues).

Lemma 11.3. *Suppose $t \in \mathbb{R}$, $u \in U^+(x)$.*

(a) $g_t\mathcal{C}_{ij}[x] \cap B[g_t x] \cap g_t B[x] = \mathcal{C}_{ij}[g_t x] \cap B[g_t x] \cap g_t B[x]$.
(b) $u\mathcal{C}_{ij}[x] \cap B[ux] \cap uB[x] = \mathcal{C}_{ij}[ux] \cap B[ux] \cap uB[x]$.

Proof. Note that the sets $U^+[x]$ and $\mathbf{E}_{[ij],bdd}(x)$ are g_t-equivariant. Therefore, so are the $\mathcal{E}_{ij}[x]$, which implies (a). Part (b) is also clear, since locally, by Lemma 8.2, $(u)_*\mathbf{E}_{ij}(x) = \mathbf{E}_{ij}(ux)$. $\square$

The measures ξ_{ij}. We now define $\xi_{ij}(x)$ to be the conditional measure of ν along the $\mathcal{C}_{ij}[x]$. In other words, $\xi_{ij}(x)$ is defined so that for any measurable $\phi : X \to \mathbb{R}$,

$$E(\phi \mid \mathcal{C}_{ij})(x) = \int_X \phi \, d\xi_{ij}(x).$$

We view $\xi_{ij}(x)$ is a measure $W^+[x]$ which is supported on $\mathcal{C}_{ij}[x]$.

The measures f_{ij}. We can identify $W^+[x]$ with the vector space $W^+(x) \cong \mathbb{R}^n$, where x corresponds to the origin in $\mathbb{R}^n$. Let $f_{ij}(x)$ be the pullback to $\mathbb{R}^n$ of $\xi_{ij}(x)$ under this identification. We will also call the $f_{ij}(x)$ conditional measures. (The term "leaf-wise" measures is used in [EL] in a related context). We abuse notation slightly and write formulas such as

$$E(\phi \mid \mathcal{C}_{ij})(x) = \int_X \phi \, df_{ij}(x).$$

The main property of the measures f_{ij} which we will need is the following:

Proposition 11.4. *There exists $0 < \alpha_0 < 1$ depending only on the Lyapunov spectrum, and for every $\delta > 0$ there exists a compact set K_0 with $\nu(K_0) > 1 - \delta$ such that the following holds: Suppose $0 < \epsilon < 1$, $C < \infty$, $t > 0$, $t' > 0$, and $|t' - t| < C$. Furthermore suppose $q \in K_0$ and $q' \in W^-[q] \cap K_0$ satisfy (5.6) and (5.7). Let $q_1 = g_\ell q$, $q_1' = g_\ell q'$. Also let $q_3 = g_{t'}^{ij} q_1$, $q_3' = g_{t'}^{ij} q_1'$. Suppose q_1, q_1', q_3, q_3' all belong to K_0.*

Suppose $u \in U^+(q_1)$, $u' \in U^+(q_1')$, and let $q_2 = g_t^{ij} u q_1$. We write $q_2 = g_\tau u q_1$ for some $\tau > 0$, and let $q_2' = g_\tau u' q_1'$ (see Figure 1). Also suppose $C^{-1}\epsilon \le d(q_2, q_2') \le C\epsilon$ and $\ell > \alpha_0 \tau$.

In addition, suppose there exist $\tilde{q}_2$ and $\tilde{q}_2' \in \mathcal{E}_{ij}[\tilde{q}_2]$ (so in particular $\tilde{q}_2' \in W^+[\tilde{q}_2]$) such that $d(\tilde{q}_2, q_2) < \xi$ and $d(\tilde{q}_2', q_2') < \xi$. Suppose further that q_2, q_2', $\tilde{q}_2$ and $\tilde{q}_2'$ all belong to K_0.

Then, there exists $\xi''' > 0$ (depending on ξ, K_0 and C and t) with $\xi''' \to 0$ as $\xi \to 0$ and $t \to \infty$ such that

$$d(P^+(\tilde{q}_2, \tilde{q}_2') f_{ij}(\tilde{q}_2), f_{ij}(\tilde{q}_2')) \le \xi'''. \tag{11.4}$$

Here $d(\cdot, \cdot)$ is any metric on measures on W^+ which induces the topology of weak- convergence on the domain of common definition of the measures, up to normalization.*

Proposition 11.4 is proved in §11.2*. We give an outline of the argument below.

Outline of the proof of Proposition 11.4. The initial intuition behind the proof of Proposition 11.4 linear maps is that "one goes from q_3' to q_2' by nearly the same linear map as from q_3 to q_2; since this map is bounded on the relevant subspaces, $f_{ij}(q_2)$ should be related to $f_{ij}(q_3)$ and $f_{ij}(q_2')$ should be related to $f_{ij}(q_2)$; since $f_{ij}(q_3)$ and $f_{ij}(q_3')$ are close, $f_{ij}(q_2')$ should be related to $f_{ij}(q_2)$."

There are several problems with this argument. First, because of the need to change transversals, there is no linear map from the space $\mathrm{GSpc}(q_3)$ of generalized subspaces near q_3 to the space $\mathrm{GSpc}(q_2)$ of generalized subspaces near q_2. This difficulty is easily handled by working instead with the maps $R(q_3, q_2) : \mathbf{H}(q_3) \to \mathbf{H}(q_2)$ and $R(q_3', q_2') : \mathbf{H}(q_3') \to \mathbf{H}(q_2')$.

The second difficulty is connected to the first. We would like to say that the two maps $R(q_3, q_2)$ and $R(q_3, q_2')$ are close, but the domains and ranges of the maps are different. Thus we need "connecting" linear maps from $\mathbf{H}(q_3)$ to $\mathbf{H}(q_3')$, and also from

$\mathbf{H}(q_2)$ to $\mathbf{H}(q_2')$. The first map is easy to construct: since q_3 and q_3' are in the same leaf of W^-, we can just use the map $\mathbf{P}^-(q_3, q_3')$ induced by the "W^--connection map" $P^-(q_3, q_3')$ defined in §4.2.

Instead of constructing directly a map from $\mathbf{H}(q_2)$ to $\mathbf{H}(q_2')$ we construct maps $\mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) : \mathbf{H}(q_2) \to \mathbf{H}(\tilde{q}_2)$ and $\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') : \mathbf{H}(q_2') \to \mathbf{H}(\tilde{q}_2')$. Since q_2 and $\tilde{q}_2$ are close, and also since q_2' and $\tilde{q}_2'$ are close, these maps are in a suitable sense close to the identity. Then, since $\tilde{q}_2$ and $\tilde{q}_2'$ are on the same leaf of W^+, we have the map $\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')$ induced by the W^+-connection map $P^+(\tilde{q}_2, \tilde{q}_2')$ of §4.2.

Thus, finally we have two maps from $\mathbf{H}(q_3)$ to $\mathbf{H}(\tilde{q}_2')$:

$$\mathbf{A} = \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)$$

and

$$\mathbf{A}' = \mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') \circ R(q_3', q_2') \circ \mathbf{P}^-(q_3, q_2)$$

Even though $\mathbf{A}$ and $\mathbf{A}'$ are defined on $\mathbf{H}(q_3)$, in what follows we only need to consider their restrictions to $\mathbf{E}_{[ij],bdd}(q_3) \subset \mathbf{H}(q_3)$; we will denote the restrictions by $\mathbf{B}$ and $\mathbf{B}'$ respectively.

We would like to show that $\mathbf{B}$ and $\mathbf{B}'$ are close. By linearity, it is enough to show that the restrictions of $\mathbf{B}$ and $\mathbf{B}'$ to each $\mathbf{E}_{ij,bdd}(q_3) \subset \mathbf{E}_{[ij],bdd}(q_3)$ are close. Note that by Proposition 4.4 (a), $\mathbf{P}^-(q_3, q_3')\mathbf{E}_{ij,bdd}(q_3) = \mathbf{E}_{ij,bdd}(q_3')$. Continuing this argument, we see that the two subspaces $\mathbf{B}\,\mathbf{E}_{ij,bdd}(q_3)$ and $\mathbf{B}'\,\mathbf{E}_{ij,bdd}(q_3)$ are close to $\mathbf{E}_{ij,bdd}(\tilde{q}_2')$ (and thus are close to each other). Also, from the construction and Proposition 10.2, we see that both $\mathbf{B}$ and $\mathbf{B}'$ are uniformly bounded linear maps. However, this is still not enough to conclude that $\mathbf{B}$ and $\mathbf{B}'$ are close. In fact we also check that $\mathbf{B}$ and $\mathbf{B}'$ are close modulo $\mathbf{V}_{i-1}(\tilde{q}_2)$. (This part of the argument uses the assumptions on q, q', q_1, q_1', etc). Since $\mathbf{E}_{ij,bdd}(\tilde{q}_2)$ and $\mathbf{V}_{i-1}(\tilde{q}_2)$ are transverse, this is enough to show

$$\|\mathbf{B} - \mathbf{B}'\| \to 0 \text{ as } \xi \to 0. \tag{11.5}$$

The final part of the proof of Proposition 11.4 consists of deducing (11.4) from (11.5) and the fact that $\mathbf{B}$ and $\mathbf{B}'$ are uniformly bounded (Proposition 10.2).

11.1*. Proof of Lemma 11.2.

Lemma 11.5. *Suppose $K \subset X$ with $\nu(K) > 1 - \delta$. Then, for any $\theta' > 0$ there exists a subset $K^* \subset X$ with $\nu(K^*) > 1 - 2\kappa^2\delta/\theta'$ such that for any $x \in K^*$ and any $\ell > 0$,*

$$|\mathcal{F}_{ij}[x, \ell] \cap K| > (1 - \theta')|\mathcal{F}_{ij}[x, \ell]|. \tag{11.6}$$

Proof. For $t > 0$ let

$$\mathcal{B}_t[x] = g_{-t}(J[g_t x] \cap U^+[x]) = g_{-t}\mathcal{B}[g_t x].$$

Let $s > 0$ be arbitrary. Let $K_s = g_{-s}^{ij} K$. Then $\nu(K_s) > 1 - \kappa\delta$. Then, by Lemma 3.11, there exists a subset K_s' with $\nu(K_s') \geq (1 - 2\kappa\delta/\theta')$ such that for $x \in K_s'$ and all $t > 0$,

$$|K_s \cap \mathcal{B}_t[x]| \geq (1 - \theta'/2)|K_s|.$$

Let $K_s^* = g_s^{ij} K_s'$, and note that $g_s^{ij}\mathcal{B}_t[x] = \mathcal{F}_{ij}[g_s x, s-t]$. Then, for all $x \in K_s^*$ and all $0 < s-t < s$,

$$|\mathcal{F}_{ij}[x, s-t] \cap K| \geq (1-\theta'/2)|\mathcal{F}_{ij}[x, s-t]|.$$

We have $\nu(K_s^*) \geq (1 - 2\kappa^2\delta/\theta')$. Now take a sequence $s_n \to \infty$, and let K^* be the set of points which are in infinitely many $K_{s_n}^*$. □

Proof of Lemma 11.2. Let $\theta_1 > 0$ and $\delta > 0$ be small constants to be chosen later. Let $K \subset X$ and $C > 0$ be such that $\nu(K) > 1-\delta$, for $x \in K$ the Lemma 6.5 (b) holds with $c_1(x) > C^{-1}$, and for all $x \in K$, all $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$ and all $\ell > 0$, for at least $(1-\theta_1)$ fraction of $y \in \mathcal{F}_{ij}[x,\ell]$,

$$\|R(x,y)\mathbf{v}\| < C\|\mathbf{v}\|. \tag{11.7}$$

By Lemma 11.5 there exists a subset $K^* \subset K$ with $\nu(K^*) \geq (1-2\kappa^2\delta^{1/2})$ such that for $x \in K^*$, (11.6) holds with $\theta' = \delta^{1/2}$. Furthermore, we may ensure that for $x \in K^*$, $K^* \cap \mathcal{F}_{ij}[x]$ is relatively open in $\mathcal{F}_{ij}[x]$. (Indeed, suppose $z \in \mathcal{F}_{ij}[x]$ is near $x \in K^*$. Then, there exists ℓ_0 such that for $\ell > \ell_0$, $\mathcal{F}_{ij}[x,\ell] = \mathcal{F}_{ij}[z,\ell]$ and thus (11.6) holds for z. For $\ell < \ell_0$, (11.6) holds for z sufficiently close to x by continuity.) Let

$$\Psi = \{x \in X \ : \ \lim_{T\to\infty} |\{t \in [0,T] \ : \ g_{-t}x \in K^*\}| \geq (1-2\kappa^2\delta^{1/2}).$$

Then $\nu(\Psi) = 1$. From its definition, Ψ is invariant under g_t. Since $K^* \cap \mathcal{F}_{ij}[x]$ is relatively open in $\mathcal{F}_{ij}[x]$, Ψ is saturated by the leaves of $\mathcal{F}_{ij}$. This implies that Ψ is (locally) invariant under U^+. Now, let

$$K_N = \{x \in \Psi \ : \quad \text{for all } T > N, \quad |\{t \in [0,T] \ : \ g_{-t}x \in K^*\}| \geq (1-4\kappa^2\delta^{1/2})T\}.$$

(We may assume that $4\kappa^2\delta^{1/2} \ll 1$.) We have $\bigcup_N K_N = \Psi$.

Suppose $x_0 \in K_N$, $x_1 \in K_N$, with $d(x_0,x_1) < 1$. For $k = 0,1$, let $\mathcal{Q}_k \subset \mathcal{E}_{ij}[x_k]$ be such that

$$hd_{x_k}(\mathcal{Q}_k, U^+[x_k]) \leq 1,$$

and the vector

$$\mathbf{v}_k = \mathbf{j}(\mathcal{U}_{x_k}^{-1}(\mathcal{Q}_{1-k}))$$

satisfies $\|\mathbf{v}_k\| \leq 1$.

We claim that $\mathbf{v}_k \in \mathbf{H}(x_k)$. Indeed, we may write $\mathcal{U}_{x_{1-k}}^{-1}(\mathcal{Q}_{1-k}) = (M_{1-k}, v_{1-k})$. Also we may write $\mathcal{U}_{x_k}^{-1}(U^+[x_{1-k}]) = (M_k', v_k')$. Then, $\mathcal{Q}_{1-k}$ is parametrized (from x_k) by a pair (M_k'', w_k) where $w_k \in W^+(x_k)$, and

$$M_k'' = (I + M_{1-k}) \circ (I + M_k') - I$$

(This parametrization is not necessarily adapted to $Z(x_k)$.) Since M_{1-k} and M_k' are both in $\mathcal{H}_+$, $M_k'' \in \mathcal{H}_+(x_k)$. Thus, $\mathbf{v}_k = \mathbf{S}_{x_k}(\mathbf{j}(M_k'', w_k)) \in \mathbf{H}(x_k)$.

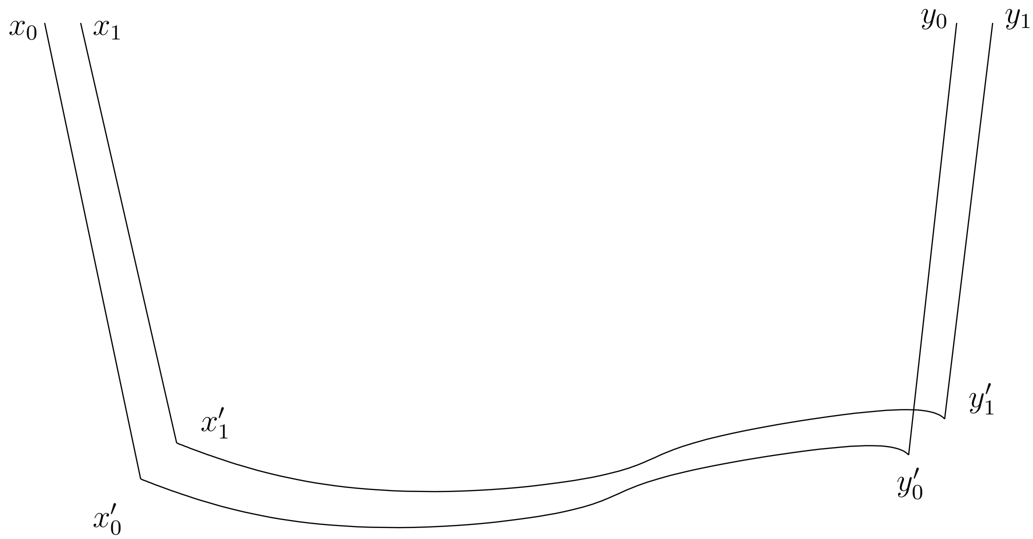

Figure 4. Proof of Lemma 11.2

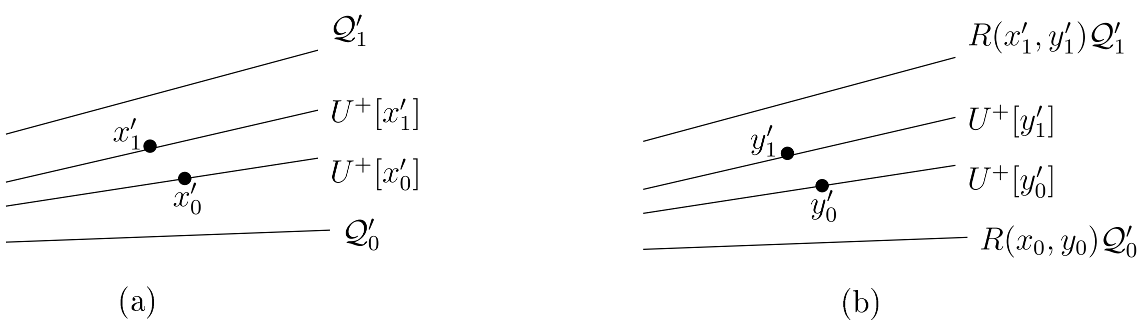

Figure 5. Proof of Lemma 11.2

In (b), the subspaces $U^+[y'_0]$ and $U^+[y'_1]$ stay close since $x'_1 \in \mathcal{E}_{ij}(x'_0)$, and also for $k \in \{0, 1\}$, the subspaces $R(x'_k, y'_k)\mathcal{Q}'_k$ and $U^+[y'_k]$ stay close since $\mathcal{Q}'_k \in \mathcal{E}_{ij,bdd}(x'_k)$.

For $C_1(N)$ sufficiently large, we can find $C_1(N) < t < 2C_1(N)$ such that $x'_0 \equiv g^{ij}_{-t} x_0 \in K^*$, $x'_1 \equiv g^{ij}_{-t} x_1 \in K^*$. Let $\mathbf{v}'_k = g^{ij}_{-t}\mathbf{v}_k$, $\mathcal{Q}'_k = g^{ij}_{-t}\mathcal{Q}_k$. By choosing $C_1(N)$ sufficiently large (depending on N), we can ensure that

$$hd_{x'_k}(U^+[x'_k], U^+[x'_{1-k}]) \le C^{-3}, \qquad hd_{x'_k}(\mathcal{Q}'_k, U^+[x'_k]) \le C^{-3}.$$

By Lemma 6.5, since $x'_k \in K$,

$$\|\mathbf{j}(\mathcal{U}_{x'_k}^{-1}(U^+[x'_{1-k}]))\| \le C^{-2}, \qquad \|\mathbf{j}(\mathcal{U}_{x'_k}^{-1}(\mathcal{Q}'_k))\| \le C^{-2}. \tag{11.8}$$

Let $\ell > 0$ be arbitrary, and let ℓ' be such that $g_t^{ij}\mathcal{F}_{ij}[x,\ell'] = \mathcal{F}_{ij}[x,\ell]$. Then, for $k = 0,1$, since $x'_k \in K^*$,

$$|\{y'_k \in \mathcal{F}_{ij}[x'_k,\ell'] \ : \ y'_k \in K\}| \geq (1-\delta^{1/2})|\mathcal{F}_{ij}[x',\ell']|,$$

Since $U^+[x_1] \in \mathcal{E}_{ij}[x_0]$, we have $U^+[x'_1] \in \mathcal{E}_{ij}[x'_0]$, and thus $\mathbf{j}(\mathcal{U}_{x'_0}^{-1}(U^+[x'_1])) \in \mathbf{E}_{[ij],bdd}(x'_0)$. Since $x'_0 \in K$, we have by (11.7), for at least $(1-\theta_1)$-fraction of $y'_0 \in \mathcal{F}_{ij}[x'_0,\ell']$,

$$\|R(x'_0,y'_0)\mathbf{j}(\mathcal{U}_{x'_0}^{-1}(U^+[x'_1]))\| \leq C\|\mathbf{j}(\mathcal{U}_{x'_0}^{-1}(U^+[x'_1]))\| \leq C^{-1}, \tag{11.9}$$

where we have used (11.8) for the last estimate. Let $\theta'' = 2\theta_1 + 2\delta^{1/2}$. Then, for at least $1-\theta''/2$ fraction of $y'_0 \in \mathcal{F}_{ij}[x'_0,\ell']$, $y'_0 \in K$ and (11.9) holds. Therefore, by Lemma 6.5, for at least $(1-\theta''/2)$-fraction of $y'_0 \in \mathcal{F}_{ij}[x'_0,\ell']$, for a suitable $y'_1 \in \mathcal{F}_{ij}[x'_1,\ell']$,

$$hd_{y'_0}(\mathcal{U}^+[y'_0],\mathcal{U}^+[y'_1]) \leq 1. \tag{11.10}$$

Also, since $\mathcal{Q}_k \in \mathcal{E}_{ij}[x_k]$, $\mathcal{Q}'_k \in \mathcal{E}_{ij}[x'_k]$, and thus $\mathbf{j}(\mathcal{U}_{x'_k}^{-1}(\mathcal{Q}'_k)) \in \mathbf{E}_{[ij],bdd}(x'_k)$. Hence, by (11.7), for at least $(1-\theta)$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k,\ell']$,

$$\|R(x'_k,y'_k)\mathbf{j}(\mathcal{U}_{x'_k}^{-1}(\mathcal{Q}'_k))\| \leq C\|\mathbf{j}(\mathcal{U}_{x'_k}^{-1}(\mathcal{Q}'_k))\| \leq C^{-1}. \tag{11.11}$$

where we used (11.8) for the last estimate. Then, for at least $(1-\theta''/2)$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k,\ell']$, $y'_k \in K$ and (11.11) holds. Therefore, by Lemma 6.5, for at least $(1-\theta''/2)$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k,\ell']$,

$$hd_{y'_k}(U^+[y'_k], R(x'_k,y'_k)\mathcal{Q}'_k) \leq 1.$$

Therefore, by (11.10), for at least $(1-\theta'')$-fraction of $y'_k \in \mathcal{F}_{ij}[x'_k,\ell']$, for a suitable $y'_{1-k} \in \mathcal{F}_{ij}[x'_{1-k},\ell']$,

$$hd_{y'_k}(U^+[y'_k], R(x'_{1-k},y'_{1-k})\mathcal{Q}'_{1-k}) \leq 2. \tag{11.12}$$

Let

$$\mathbf{w}'_k = \mathbf{j}(\mathcal{U}_{y'_k}^{-1}(R(x'_{1-k},y'_{1-k})\mathcal{Q}'_{1-k})) = R(x'_k,y'_k)\mathbf{v}'_k.$$

Then, assuming $y'_0 \in K$ and (11.12) holds, by Lemma 6.5,

$$\|\mathbf{w}'_k\| \leq C.$$

Let $y_k = g_t^{ij}y'_k$, and let

$$\mathbf{w}_k = R(y'_k,y_k)\mathbf{w}'_k = R(x_k,y_k)\mathbf{v}_k.$$

Then, for at least $(1-\theta'')$-fraction of $y_k \in \mathcal{F}_{ij}[x_k,\ell]$, $\|R(x_k,y_k)\mathbf{v}_k\| \leq C_2(N)$. This implies, by Proposition 10.3, that $\mathbf{v}_k \in \mathbf{E}_{[ij],bdd}(x_k)$. (By making $\theta_1 > 0$ and $\delta > 0$ sufficiently small, we can make sure that $\theta'' < \theta$ where $\theta > 0$ is as in Proposition 10.3.)

Thus, for all $\mathcal{Q}_k \in \mathcal{E}_{ij}[x_k]$ such that $\mathbf{j}(\mathcal{U}_{x_{1-k}}^{-1}(\mathcal{Q}_k)) \leq 1$, we have $\mathbf{j}(\mathcal{U}_{x_{1-k}}^{-1}(\mathcal{Q}_k)) \in \mathbf{E}_{[ij],bdd}(x_{1-k})$. Since both $\mathcal{U}_{x_{1-k}}^{-1}$ and $\mathbf{j}$ are analytic, this implies that $\mathbf{j}(\mathcal{U}_{x_{1-k}}^{-1}(\mathcal{Q}_k)) \in \mathbf{E}_{[ij],bdd}(x_{1-k})$ for all $\mathcal{Q}_k \in \mathcal{E}_{ij}[x_k]$. Thus, for $k=0,1$, $\mathcal{E}_{ij}[x_k] \subset \mathcal{E}_{ij}[x_{1-k}]$. This implies that $\mathcal{E}_{ij}[x_0] = \mathcal{E}_{ij}[x_1]$. $\square$

11.2*. **Proof of Proposition 11.4.** Let $\mathcal{O} \subset X$ be an open set contained in the fundamental domain, and let $x \to u_x \in U^+(x)$ be a function which is constant on each set of the form $U^+[x] \cap \mathcal{O}$. Let $T_u : \mathcal{O} \to X$ be the map which takes $x \to u_x x$.

Lemma 11.6. *Suppose $E \subset \mathcal{O}$. Then $\nu(T_u(E)) = \nu(E)$.*

Proof. Without loss of generality, we may assume that $T_u(\mathcal{O}) \cap \mathcal{O} = \emptyset$. For each $x \in \mathcal{O}$, let $\tilde{U}[x]$ be a finite piece of $U^+[x]$ which contains both $U[x] \cap \mathcal{O}$ and $T_u(U[x] \cap \mathcal{O})$. We may assume that $\tilde{U}[x]$ is the same for all $x \in U[x] \cap \mathcal{O}$. Let $\tilde{\mathcal{U}}$ be the σ-algebra of functions which are constant along each $\tilde{U}[x]$. Then, for any measurable $\phi : X \to \mathbb{R}$,

$$\int_X \phi \, d\nu = \int_X \mathbb{E}(\phi \mid \tilde{\mathcal{U}}) \, d\nu$$

Now suppose ϕ is supported on $\mathcal{O}$. We have $\mathbb{E}(\phi \circ T_u \mid \tilde{\mathcal{U}}) = \mathbb{E}(\phi \mid \tilde{\mathcal{U}})$ since the conditional measures along U^+ are Haar, and T_u restricted to $\mathcal{O} \cap U^+[x]$ is a translation. Thus

$$\int_X \phi \circ T_u \, d\nu = \int_X \mathbb{E}(\phi \circ T_u \mid \mathcal{U}) \, d\nu = \int_X \mathbb{E}(\phi \mid \tilde{\mathcal{U}}) \, d\nu = \int_X \phi \, d\nu.$$

□

We also recall the following standard fact:

Lemma 11.7. *Suppose $\Psi : X \to X$ preserves ν, and also for almost all x, $\mathcal{C}_{ij}[\Psi(x)] \cap B[\Psi(x)] \cap \Psi(B[x]) = \Psi(\mathcal{C}_{ij}[x]) \cap B[\Psi(x)] \cap \Psi(B[x])$. Then,*

$$f_{ij}(\Psi(x)) \propto \Psi_* f_{ij}(x),$$

in the sense that the restriction of both measures to the set $B[\Psi(x)] \cap \Psi(B[x])$ where both make sense is the same up to normalization.

Proof. See [EL, Lemma 4.2(iv)]. □

Lemma 11.8. *We have (on the set where both are defined):*

$$f_{ij}(g_t T_u g_{-s} x) \propto (g_t T_u g_{-s})_* f_{ij}(x).$$

Proof. This follows immediately from Lemma 11.6 and Lemma 11.7. □

The maps ϕ_x. We have the map $\phi_x : W^+(x) \to \mathcal{H}_+(x) \times W^+(x)$ given by

$$\phi_x(z) = \mathcal{U}_x^{-1}(U^+[z]). \tag{11.13}$$

(Here $\mathcal{U}_x^{-1}$ is defined using the transversal $Z(x)$.)

Suppose $Z(x)$ is an admissible transversal to $U^+(x)$. Since $f_{ij}(x)$ is Haar along U^+, we can recover $f_{ij}(x)$ from its restriction to $Z(x)$. More precisely, the following holds:

Let $\pi_2 : \mathcal{H}^+(x) \times W^+(x) \to W^+(x)$ be projection onto the second factor. Then, for $z \in Z(x)$, $\pi_2(\phi(z)) = z$. Now, suppose Z' is another transversal to $U^+(x)$. Then,

$$(f_{ij} \mid_{Z'})(x) = (\pi_2 \circ S_x^{Z'} \circ \phi)_* (f_{ij} \mid_{Z(x)}).$$

The measures $\mathbf{f}_{ij}(x)$. Let

$$\mathbf{f}_{ij}(x) = (\mathbf{j} \circ \phi_x)_* f_{ij}(x).$$

Then, $\mathbf{f}_{ij}(x)$ is a measure on $\mathbf{H}(x)$.

Lemma 11.9. *For $y \in \mathcal{F}_{ij}[x]$, we have (on the set where both are defined),*

$$\mathbf{f}_{ij}(y) \propto R(x,y)_* \mathbf{f}_{ij}(x).$$

Proof. Suppose $t > 0$ is such that $x' = g^{ij}_{-t}x$ and $y' = g^{ij}_{-t}y$ satisfy $y' \in \mathcal{B}[x']$. Let $Z[x'] = g^{ij}_{-t}Z[x]$, and let $Z[y'] = g^{ij}_{-t}Z[y]$. For $z \in Z[x']$ near x' let u_z be such that $u_z z \in Z[y']$. We extend the function $z \to u_z$ to be locally constant along U^+ in a neighborhood of $Z[x']$. Then, let

$$\Psi = g^{ij}_t \circ T_u \circ g^{ij}_{-t}.$$

Note that Ψ takes $Z[x]$ into $Z[y]$, and by Lemma 11.8,

$$\Psi_* f_{ij}(x) = f_{ij}(y). \tag{11.14}$$

By the definition of u_* in §6, for $z \in Z[x]$,

$$(R(x,y) \circ \mathbf{j} \circ \mathcal{U}_x^{-1})U^+[z] = (\mathbf{j} \circ \mathcal{U}_y^{-1})U^+[\Psi(z)].$$

Hence, by (11.13),

$$(R(x,y) \circ \mathbf{j} \circ \phi_x)(z) = (\mathbf{j} \circ \phi_y \circ \Psi)(z), \tag{11.15}$$

where ϕ_y is relative to the transversal $Z(y)$ and ϕ_x is relative to the transversal $Z(x)$. (Here we have used the fact that $\Psi(U^+[z]) = U^+[\Psi(z)]$ which follows from the equivariance of U^+. Also, in (11.15), $R(x,y)$ is as in §9.3.) Now the lemma follows from (11.14) and (11.15). $\square$

Let $P^+(x,y)$ and $P^-(x,y)$ be as in §4.2. The maps $P^+(x,y)_* : \mathrm{Lie}(\mathcal{G}_+)(x) \to \mathrm{Lie}(\mathcal{G}^+)(y)$ (where we use the notation (6.9)) are an equivariant measurable flat W^+-connection on the bundle $\mathrm{Lie}(\mathcal{G}_+)$ satisfying (4.7). Then, by Proposition 4.4(a),

$$P^+(x,y)_* \operatorname{Lie}(U^+)(x) = \operatorname{Lie}(U^+)(y). \tag{11.16}$$

The maps $\mathbf{P}^+(x,y)$ and $\mathbf{P}^-(x,y)$. In view of (11.16), the maps $P^+(x,y)$ naturally induce a linear map (which we denote by $\tilde{\mathbf{P}}^+(x,y)$) from $\tilde{\mathbf{H}}(x)$ to $\tilde{\mathbf{H}}(y)$, so that for $(M,v) \in \mathcal{H}_+(x)$,

$$\mathbf{P}^+(x,y) \circ \mathbf{j}(M,v) = \mathbf{j}(P^+(x,y) \circ M \circ P^+(x,y)^{-1}, P^+(x,y)v).$$

Let $\mathbf{P}^+(x,y) = \mathbf{S}_y^{Z(y)} \circ \tilde{\mathbf{P}}^+(x,y)$. Then the maps $\mathbf{P}^+(x,y) : \mathbf{H}(x) \to \mathbf{H}(y)$ are an equivariant measurable flat W^+-connection on the bundle $\mathbf{H}$ satisfying (4.7). Then, by Proposition 4.4(a), we have

$$\mathbf{P}^+(x,y)\mathbf{E}_{ij,bdd}(x) = \mathbf{E}_{ij,bdd}(y). \tag{11.17}$$

For $y \in W^-[x]$, we have a map $\mathbf{P}^-(x,y)$ with analogous properties.

The maps $P^Z(x,y)$ and $\mathbf{P}^Z(x,y)$. We also need to define a map between $\mathbf{H}(x)$ and $\mathbf{H}(y)$ even if x and y are not on the same leaf of W^+ or W^-. For every $v_i \in \mathcal{V}_i(x)$, and $i \in \Lambda$ (where Λ is the Lyapunov spectum) we can write

$$v_i = v_i' + v_i'' \quad v_i' \in \mathcal{V}_i(y), \quad v_i'' \in \bigoplus_{j \neq i} \mathcal{V}_i(y).$$

Let $P^\sharp(x,y) : W^+(x) \to W^+(y)$ be the linear map whose restriction to $\mathcal{V}_i(x)$ sends v_i to v_i'. By definition, $P^\sharp(x,y)$ sends $\mathcal{V}_i(x)$ to $\mathcal{V}_i(y)$, but it is not clear that $P^\sharp(x,y)_* \operatorname{Lie}(U^+)(x) = \operatorname{Lie}(U^+)(y)$. To correct this, given a Lyapunov-adapted transversal $Z(x)$, we let $M(x,y) : \operatorname{Lie}(\mathcal{G}_+)(x) \to \operatorname{Lie}(\mathcal{G}_+)(y)$ be the linear map such that

$$(I + M(x,y)) \operatorname{Lie}(U^+)(x) = P^\sharp(x,y)_*^{-1} \operatorname{Lie}(U^+)(y),$$

and $M(x,y)\mathcal{V}_i(\operatorname{Lie}(U^+))(x) \subset Z_i(x)$, where $Z_i(x) = Z(x) \cap \mathcal{V}_i(\operatorname{Lie}(\mathcal{G}_+))(x)$ is as in §6. Then, let

$$P^{Z(x)}(x,y) = P^\sharp(x,y)_* \circ M(x,y).$$

Then, if $Z = Z(x)$ is Lyapunov adapted, we have

$$P^{Z(x)}(x,y)\mathcal{V}_i(\operatorname{Lie}(G_+))(x) = \mathcal{V}_i(\operatorname{Lie}(\mathcal{G}_+))(y) \text{ and } P^{Z(x)} \operatorname{Lie}(U^+)(x) = \operatorname{Lie}(U^+)(y).$$

Then $P^{Z(x)}$ gives a map $\mathcal{H}_+(x) \to \mathcal{H}_+(y)$ sending $f \in \mathcal{H}_+(x)$ to $P^{Z(x)}(x,y) \circ f$. Therefore, (after possibly composing with a change in transversal map $\mathbf{S}$) $P^{Z(x)}(x,y)$ induces a map we will call $\mathbf{P}^{Z(x)}(x,y)$ between $\mathbf{H}(x)$ and $\mathbf{H}(y)$. This map has the equivariance property

$$\mathbf{P}^{g_{-t}Z(x)}(g_{-t}x, g_{-t}y) = g_{-t} \circ \mathbf{P}^{Z(x)}(x,y) \circ g_t.$$

Lemma 11.10. *Suppose x and $y \in X$, and $s > 0$ are such that for all $|t| < s$, $d(g_t x, g_t y) = O(1)$, and also $x \in K_{thick}$ where K''_{thick} is as in Lemma 3.4. Then, there exists $\alpha > 0$ depending only on the Lyapunov spectrum such that for all i,*

$$d(\mathcal{V}_i(x), \mathcal{V}_i(y)) = O(e^{-\alpha s}).$$

Proof. Let $v \in \mathcal{V}_i(x)$ be a unit vector. We can write

$$v = v_+ + v_0 + v_-,$$

where $v_0 \in \mathcal{V}_i(y)$, $v_+ \in \bigoplus_{j>i} \mathcal{V}_j(y)$, $v_- \in \bigoplus_{j<i} \mathcal{V}_j(y)$. Then, flowing forward and using Lemma 3.4 shows that $\|v_-\| = O(e^{-\alpha s})$, and flowing backward and using Lemma 3.4 shows that $\|v_+\| = O(e^{-\alpha s})$. $\square$

For every $\delta > 0$ and every $0 < \alpha < 1$ there exist compact sets $K_0 \subset K^\sharp \subset X$ with $\nu(K_0) > 1 - \delta$ such that the following hold:

($K^\sharp 1$) The functions $U^+(x)$ and $\mathcal{V}_i(x)$ are uniformly continuous on $K^\sharp$.

($K^\sharp 2$) The functions $Z(x)$ are uniformly continuous on $K^\sharp$.

($K^\sharp 3$) The functions $\mathbf{E}_{ij,bdd}(x)$ are uniformly continuous on $K^\sharp$.

($K^\sharp 4$) The functions $f_{ij}(x)$ and $\mathbf{f}_{ij}(x)$ are uniformly continuous on $K^\sharp$ (in the weak-* convergence topology).

($K^\sharp 5$) There exists $t_0 > 0$ and $\epsilon' < 0.25\alpha \min|\lambda_i - \lambda_j|$ such that for $t > t_0$, $x \in K^\sharp$, all i, and any $v \in \mathcal{V}_i(x)$,

$$e^{(\lambda_i - \epsilon')t}\|v\| \leq \|(g_t)_* v\| \leq e^{(\lambda_i+\epsilon')t}\|v\|$$

($K^\sharp 6$) The function $C_3(\cdot)$ of Proposition 10.2 is uniformly bounded on $K^\sharp$.

($K^\sharp 7$) $\mathbf{E}_{ij,bdd}(x)$ and $\mathbf{V}_{i-1}(x)$ are transverse for $x \in K^\sharp$.

($K^\sharp 8$) $K^\sharp \subset K''_{thick}$ where K''_{thick} is as in Lemma 3.4 (c).

($K^\sharp 9$) For all $x \in K^\sharp$, $d(x, \partial B[x]) > c_0(\delta)$ where $B[x]$ is as in the beginning of §11, and $c_0(\delta) \to 0$ as $\delta \to 0$.

($K^\sharp 10$) There exists a constant $C_1 = C_1(\delta) < \infty$ such that for $x \in K_0$ and all $T > C_1(\delta)$ and all ij we have

$$|\{t \in [C_1, T] \ : \ g^{ij}_{-t} x \in K^\sharp\}| \geq 0.99(T - C_1).$$

Proposition 11.11. *Suppose α, ϵ, s, ℓ, t, t', q, q', β, q_1, q_1', q_3, q_3', u, u', q_2, q_2', $\tilde{q}_2$, $\tilde{q}_2'$, C, ξ are as in Proposition 11.4. Then, (assuming ϵ' in ($K^\sharp 5$) is sufficiently small depending on α_0 and the Lyapunov spectrum),*

(a) *There exists $\xi' > 0$ (depending on ξ, K_0 and C and t) with $\xi' \to 0$ as $\xi \to 0$ and $t \to \infty$ such that for $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(q_3)$,*

$$\|\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') \circ R(q_3', q_2') \circ \mathbf{P}^-(q_3, q_3')\mathbf{v} - \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)\mathbf{v}\| \leq \xi'\|\mathbf{v}\|. \tag{11.18}$$

(b) *There exists $\xi'' > 0$ (depending on ξ, K_0, C and t) with $\xi'' \to 0$ as $\xi \to 0$ and $t \to \infty$ such that*

$$d(\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')\mathbf{f}_{ij}(\tilde{q}_2), \mathbf{f}_{ij}(\tilde{q}_2')) \leq \xi''.$$

Here $d(\cdot,\cdot)$ is any metric which induces the weak- convergence topology on the domain of common definition of the measures, up to normalization.*

Proof of (a). Note that by (9.5), we have $\kappa^{-1}\tau \leq t \leq \kappa\tau$, where κ depends only on the Lyapunov spectrum. Also, by assumption we have

$$\ell > \alpha_0 \tau, \tag{11.19}$$

where α_0 depends only on the Lyapunov spectrum.

Since $\|P^-(q, q') - I\| = O(1)$, in view of ($K^\sharp 5$) and Lemma 4.2 (b) and (c) applied to P^-, we have

$$\|P^-(q_1, q_1') - I\| = O(e^{-\alpha\ell}) = O(e^{-\alpha\alpha_0\tau}) \tag{11.20}$$

where α depends only on the Lyapunov spectrum. Let $\tilde{q}_1 = g^{ij}_{-t}\tilde{q}_2 = g_{-\tau}\tilde{q}_2$, $\tilde{q}_1' = g_{-\tau}\tilde{q}_2'$. Then,

$$\tilde{q}_1' = g^{ij}_{-t''}\tilde{q}_2', \quad \text{where } |t'' - t| = O(1). \tag{11.21}$$

Since $\|P^+(\tilde{q}_2, \tilde{q}_2') - I\| = O(1)$, and in view of Lemma 4.2 (b) and (c),

$$\|P^+(\tilde{q}_1, \tilde{q}_1') - I\| = O(e^{-\alpha_1 \tau}), \tag{11.22}$$

where α_1 depends only on the Lyapunov spectrum.

Also, by Lemma 11.10, since $d(g_s u' q_1', g_s \tilde{q}_1') = O(1)$ for all s with $|s| \le \ell$,

$$\|P^\sharp(u' q_1', \tilde{q}_1') - I\| = O(e^{-\alpha_2 \ell}) = O(e^{-\alpha_2 \alpha_0 \tau}), \tag{11.23}$$

where α_2 depends only on the Lyapunov spectrum. Since we have $hd(U^+[q_2], U^+[q_2']) \approx \epsilon = O(1)$, by Lemma 3.4 (c) we have

$$hd(U^+[\tilde{q}_1'], U^+[u' q_1']) = O(e^{-\alpha_3 \tau}).$$

By the multiplicative ergodic theorem, the restriction of g_τ to $\mathcal{V}_i(W^+)$ is $e^{\lambda_i \tau} h_\tau$, where $\|h_\tau\| = O(e^{\epsilon' \tau})$. Therefore, (cf. the proof of Lemma 6.12),

$$d(g_{-\tau} Z(q_2') \cap \mathcal{V}_i(u' q_1'), U^+(u' q_1') \cap \mathcal{V}_i(u' q_1')) \ge c e^{-\epsilon' \tau}$$

where α_3 and α_4 depend only on the Lyapunov spectrum. Then, (as in the proof of Lemma 6.12),

$$\|M(u' q_1', \tilde{q}_1')\| = O(e^{\epsilon' \tau}) \tag{11.24}$$

Therefore, by (11.23) and (11.24),

$$\|P^{g_{-\tau} Z(q_2')}(u' q_1', \tilde{q}_1') - I\| = O(e^{-\alpha' \tau}), \tag{11.25}$$

where α' depends only on α_0 and the Lyapunov spectrum. In the same way,

$$\|P^{g_{-\tau} Z(q_2)}(u q_1, \tilde{q}_1) - I\| = O(e^{-\alpha' \tau}), \tag{11.26}$$

We also have $\|u - u'\| = O(e^{-\alpha \tau})$ (by Lemma 6.8). Therefore, by (11.19), (11.20), (11.25), (11.26) and (11.22), for any $\mathbf{w} \in \mathbf{E}_{ij,bdd}(q_1)$,

$$\begin{aligned}\|\mathbf{P}^{g_{-\tau} Z(q_2')}(u' q_1', \tilde{q}_1') \circ (u')_* \circ \mathbf{P}^-(q_1, q_1')\mathbf{w} - \\ \mathbf{P}^+(\tilde{q}_1, \tilde{q}_1') \circ \mathbf{P}^{g_{-\tau} Z(q_2)}(u q_1, \tilde{q}_1) \circ (u)_* \mathbf{w}\| = O(e^{-\alpha \tau} \|\mathbf{w}\|),\end{aligned}$$

where α depends only on α_0 and the Lyapunov spectrum. Hence,

$$\begin{aligned}\mathbf{P}^{g_{-\tau} Z(q_2')}(u' q_1', \tilde{q}_1') \circ (u')_* \circ \mathbf{P}^-(q_1, q_1')\mathbf{w} = \\ = \mathbf{P}^+(\tilde{q}_1, \tilde{q}_1') \circ \mathbf{P}^{g_{-\tau} Z(q_2)}(u q_1, \tilde{q}_1) \circ (u)_* \mathbf{w} + \mathbf{w}'\end{aligned} \tag{11.27}$$

where $\mathbf{w}' \in \mathbf{H}(\tilde{q}_1')$ satisfies

$$\|\mathbf{w}'\| = O(e^{-\alpha \tau} \|\mathbf{w}\|) = O_\epsilon(e^{-(\lambda_i + \alpha - \epsilon')\tau} \|\mathbf{v}\|), \tag{11.28}$$

where we wrote $\mathbf{w} = g_{-t'}^{ij} \mathbf{v}$ for some $\mathbf{v} \in \mathbf{E}_{ij}(q_3)$, and we have used $(K^\sharp 5)$ for the last estimate. We now apply $g_\tau = g_{t''}^{ij}$ (cf. (11.21)) to both sides of (11.27) and take the

quotient mod $\mathbf{V}_{i-1}(\tilde{q}_2')$. We get

$$\text{(11.29)}\quad \begin{aligned} &\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') \circ g_{t''}^{ij} \circ (u')_* \circ \mathbf{P}^-(q_1, q_1')\mathbf{w} + \mathbf{V}_{i-1}(\tilde{q}_2') = \\ &\qquad = \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ [g_{t''}^{ij} \circ (u)_* \mathbf{w} + g_{t''}^{ij}\mathbf{w}'] + \mathbf{V}_{i-1}(\tilde{q}_2'). \end{aligned}$$

We may write

$$\mathbf{w}' = \sum_k \mathbf{w}_k, \qquad \mathbf{w}_k \in \mathcal{V}_k(u'q_1').$$

Then,

$$g_{t''}^{ij}\mathbf{w}' + \mathbf{V}_{i-1}(\tilde{q}_2') = \sum_k g_{t''}^{ij}\mathbf{w}_k' + \mathbf{V}_{i-1}(\tilde{q}_2') = \sum_{k \geq i} g_{t''}^{ij}\mathbf{w}_k' + \mathbf{V}_{i-1}(\tilde{q}_2'),$$

since for $k < i$, $g_{t''}^{ij}\mathbf{w}_k' \in \mathbf{V}_{i-1}(\tilde{q}_2')$. By $(K^\sharp 5)$, for $k \geq i$,

$$g_{t''}^{ij}\mathbf{w}_k' = g_\tau \mathbf{w}_k' = O(e^{(\lambda_k + \epsilon')\tau}\|\mathbf{w}_k'\|) = O(e^{(\lambda_i + \epsilon')\tau}\|\mathbf{w}_k'\|) = O(e^{-\alpha_5\tau}\|\mathbf{v}\|),$$

using (11.28) (and choosing ϵ' sufficiently small depending on α_0 and the Lyapunov spectrum). Therefore, substituting into (11.29), we get, for $\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3)$,

$$\text{(11.30)}\quad \begin{aligned} &\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') \circ R(q_3', q_2') \circ \mathbf{P}^-(q_3, q_3')\mathbf{v} + \mathbf{V}_{i-1}(\tilde{q}_2') = \\ &\qquad = \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)\mathbf{v} + O(e^{-\alpha_5\tau}\|\mathbf{v}\|) + \mathbf{V}_{i-1}(\tilde{q}_2'). \end{aligned}$$

Since $\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3)$, we have $R(q_3, q_2)\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_2)$, and then

$$\text{(11.31)}\quad \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2)\mathbf{v} \in \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2)\mathbf{E}_{ij,bdd}(q_2).$$

By $(K^\sharp 2)$ and $(K^\sharp 3)$, since $d(q_2, \tilde{q}_2) < \xi$,

$$d(\mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2)\mathbf{E}_{ij,bdd}(q_2), \mathbf{E}_{ij,bdd}(\tilde{q}_2)) < \xi_0,$$

where $\xi_0 \to 0$ as $\xi \to 0$. Then, using (11.17),

$$\text{(11.32)}\qquad d(\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2)\mathbf{E}_{ij,bdd}(q_2), \mathbf{E}_{ij,bdd}(\tilde{q}_2')) < \xi_1.$$

where $\xi_1 \to 0$ as $\xi \to 0$. Also, by (11.17) (applied to $\mathbf{P}^-$), we have $\mathbf{P}^-(q_3, q_3')\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3')$. Then, $R(q_3', q_2') \circ \mathbf{P}^-(q_3, q_3')\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_2')$, and

$$\text{(11.33)}\qquad \mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') \circ R(q_3', q_2') \circ \mathbf{P}^-(q_3, q_3')\mathbf{v} \in \mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2')\mathbf{E}_{ij,bdd}(q_2').$$

By $(K^\sharp 2)$ and $(K^\sharp 3)$,

$$\text{(11.34)}\qquad d(\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2')\mathbf{E}_{ij,bdd}(q_2'), \mathbf{E}_{ij,bdd}(\tilde{q}_2')) < \xi_2,$$

where $\xi_2 \to 0$ as $\xi \to 0$. Note that $\mathbf{E}_{ij,bdd}(\tilde{q}_2') \cap \mathbf{V}_{i-1}(\tilde{q}_2') = \{0\}$. Now (11.18) for arbitrary $\mathbf{v} \in \mathbf{E}_{ij,bdd}(q_3)$ follows from (11.30), (11.31), (11.32), (11.33) and (11.34). The general case of (11.18) (i.e. for an arbitrary $\mathbf{v} \in \mathbf{E}_{[kr],bdd}(q_3)$) follows since $\mathbf{E}_{[kr],bdd}(q_3) = \bigoplus_{ij \in [kr]} \mathbf{E}_{ij,bdd}(q_3)$ and all the maps on the left-hand-side of (11.18) are linear.

Proof of (b). By $(K^\sharp 4)$,

$$d(\mathbf{P}^-(q_3, q_3')_* \mathbf{f}_{ij}(q_3), \mathbf{f}_{ij}(q_3')) \le \xi_1,$$

where $\xi_1 \to 0$ as and $t \to \infty$. By Lemma 11.9, $R(q_3', q_2')_* \mathbf{f}_{ij}(q_3') \propto \mathbf{f}_{ij}(q_2')$. In view of in view of condition $(K^\sharp 6)$, the assumption $|t - t'| < C$ and Proposition 10.2, that $R(q_3, q_2)$ is a linear map with norm bounded depending only on $K^\sharp$ and C. It then follows from (a) that $R(q_3', q_2')$ is also a linear map whose norm is bounded depending only on $K^\sharp$ and C. Furthermore, by $(K^\sharp 9)$ and Lemma 3.4 there exists a constant $C_2(\delta)$ such that if

$$C > t - t' > C_2(\delta), \tag{11.35}$$

then if we write $q_2 = g_t^{ij} u g_{-t'}^{ij} q_3$, then $g_t^{ij} u g_{-t'}^{ij} B[q_3] \supset B[q_2]$. Then, by Lemma 11.9,

$$\mathbf{f}_{ij}(q_2) \propto R(q_3, q_2)_* \mathbf{f}_{ij}(q_3) \quad \text{ and } \quad \mathbf{f}_{ij}(q_2') \propto R(q_3', q_2')_* \mathbf{f}_{ij}(q_3').$$

In view of $(K^\sharp 10)$, we can assume that (11.35) holds: otherwise we can replace q_3 and q_3' by $g_{-s}^{ij} q_3 \in K^\sharp$ and $g_{-s}^{ij} q_3' \in K^\sharp$ where $C_2(\delta) < s < 2C_2(\delta)$. (Without loss of generality we may assume that $C > 2C_2(\delta)$.) Hence, we have

$$d((R(q_3', q_2') \circ \mathbf{P}^-(q_3, q_3'))_* \mathbf{f}_{ij}(q_3), \mathbf{f}_{ij}(q_2')) \le \xi_2, \tag{11.36}$$

where $\xi_2 \to 0$ as $t \to \infty$. Thus, by $(K^\sharp 1)$, $(K^\sharp 2)$, $(K^\sharp 3)$,

$$d(\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') \mathbf{f}_{ij}(q_2'), \mathbf{f}_{ij}(\tilde{q}_2')) \le \xi_3,$$

where $\xi_3 \to 0$ as $\xi \to 0$ and $t \to \infty$. Hence,

$$d((\mathbf{P}^{Z(q_2')}(q_2', \tilde{q}_2') \circ R(q_3', q_2') \circ \mathbf{P}^-(q_3, q_3'))_* \mathbf{f}_{ij}(q_3), \mathbf{f}_{ij}(\tilde{q}_2')) \le \xi_4, \tag{11.37}$$

where $\xi_4 \to 0$ as $\xi \to 0$ and $t \to \infty$. Also, in view of (11.36), and since $\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')$ is a linear map whose norm is bounded depending only on $K^\sharp$,

$$d(\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{P}^{Z(q_2)}(q_2, \tilde{q}_2) \circ R(q_3, q_2))_* \mathbf{f}_{ij}(q_3), \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')_* \mathbf{f}_{ij}(\tilde{q}_2)) \le \xi_5, \tag{11.38}$$

where $\xi_5 \to 0$ as $\xi \to 0$ and $t \to \infty$. Now part (b) follows from (11.37), (11.38), and (11.18).

Proof of Proposition 11.4. Without loss of generality, and to simplify the notation, we may assume that $Z(\tilde{q}_2') = P^+(\tilde{q}_2, \tilde{q}_2') Z(\tilde{q}_2)$. (Otherwise, we can further compose with a reparametrization map at $\tilde{q}_2'$ which will not change the result). We have

$$\mathbf{f}_{ij}(\tilde{q}_2) = (\mathbf{j} \circ \phi_{\tilde{q}_2})_* f_{ij}(\tilde{q}_2)$$

and

$$\mathbf{f}_{ij}(\tilde{q}_2') = (\mathbf{j} \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2')$$

As in §6, let $P_*^+ : \mathcal{H}_+(\tilde{q}_2) \times W^+(\tilde{q}_2) \to \mathcal{H}_+(\tilde{q}_2') \times W^+(\tilde{q}_2')$ be given by

$$P_*^+(M, v) = (P^+(\tilde{q}_2, \tilde{q}_2')^{-1} \circ M \circ P^+(\tilde{q}_2, \tilde{q}_2'), P^+(\tilde{q}_2, \tilde{q}_2') v). \tag{11.39}$$

Then,

$$\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{j}(M, v) = \mathbf{j}(P_*^+(M, v)) \tag{11.40}$$

We write $A \approx_{\xi,t} B$ if $d(A,B) \to 0$ as $\xi \to 0$ and $t \to \infty$. Then, we have, by Proposition 11.11,

$$(\mathbf{j} \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') = \mathbf{f}_{ij}(\tilde{q}_2') \approx_{\xi,t} \mathbf{P}^+(\tilde{q}_2, \tilde{q}_2')_* \mathbf{f}_{ij}(\tilde{q}_2) = (\mathbf{P}^+(\tilde{q}_2, \tilde{q}_2') \circ \mathbf{j} \circ \phi_{\tilde{q}_2})_* f_{ij}(\tilde{q}_2)$$

By (11.40),

$$(\mathbf{j} \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') \approx_{\xi,t} (\mathbf{j} \circ P_*^+ \circ \phi_{\tilde{q}_2})_* f_{ij}(\tilde{q}_2).$$

Therefore,

$$(\phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') \approx_{\xi,t} (P_*^+ \circ \phi_{\tilde{q}_2})_* f_{ij}(\tilde{q}_2).$$

Let $\pi_2 : \mathcal{H}^+(x) \times W^+(x) \to W^+(x)$ be projection onto the second factor. Then, applying π_2 to both sides, we get

$$(\pi_2 \circ \phi_{\tilde{q}_2'})_* f_{ij}(\tilde{q}_2') \approx_{\xi,t} (\pi_2 \circ P_*^+ \circ \phi_{\tilde{q}_2})_* f_{ij}(\tilde{q}_2). \tag{11.41}$$

For $z \in Z(\tilde{q}_2)$, $\pi_2(\phi_{\tilde{q}_2}(z)) = z$, and thus in view of (11.39),

$$(\pi_2 \circ P_*^+ \circ \phi_{\tilde{q}_2})(z) \approx_{\xi,t} P^+(\tilde{q}_2, \tilde{q}_2')z. \tag{11.42}$$

By assumption, we have $Z(\tilde{q}_2') = P^+(\tilde{q}_2, \tilde{q}_2')Z(\tilde{q}_2)$. Then, similarly, for $z \in Z(\tilde{q}_2') = P^+(\tilde{q}_2, \tilde{q}_2')Z(\tilde{q}_2)$,

$$(\pi_2 \circ \phi_{\tilde{q}_2'})(z) = z. \tag{11.43}$$

Since $f_{ij}(x)$ is Haar along U^+, we can recover $f_{ij}(\tilde{q}_2)$ from its restrictions to $Z(\tilde{q}_2)$ and $f_{ij}(\tilde{q}_2')$ from its restriction to $Z(\tilde{q}_2')$. It now follows from (11.41), (11.42) and (11.43) that

$$f_{ij}(\tilde{q}_2') \approx_{\xi,t} P^+(\tilde{q}_2, \tilde{q}_2')_* f_{ij}(\tilde{q}_2).$$

□

11.3. **Dense subgroups of Nilpotent Groups.** Let N be a nilpotent Lie group. For a subgroup $\Gamma \subset N$, let $\bar{\Gamma}$ denote the topological closure of Γ, and let $\bar{\Gamma}^0$ denote the connected component $\bar{\Gamma}^0$ containing the identity e of N. Let $B(x,\epsilon)$ denote the ball of radius ϵ centered at x in some left-invariant metric on N.

Lemma 11.12. *Suppose N is a (nilpotent) Lie group, and $S \subset N$ is an (infinite) subset. For $\epsilon > 0$, let Γ_ϵ denote the subgroup generated by $S \cap B(e,\epsilon)$. Then there exists $\epsilon_1 > 0$ and a connected closed Lie subgroup N_1 of N such that for $\epsilon < \epsilon_1$, $\overline{\Gamma}_\epsilon = N_1$.*

Proof. Let $\epsilon > 0$ be arbitrary. Since we have $\bar{\Gamma}^0_{\epsilon'} \subset \bar{\Gamma}^0_\epsilon$ for $\epsilon' < \epsilon$, there exists $\epsilon_0 > 0$ such that for $\epsilon \le \epsilon_0$, $\dim \bar{\Gamma}^0_\epsilon$ (and thus $\bar{\Gamma}^0_\epsilon$ itself) is independent of ϵ. Thus there exists a connected closed subgroup $N_1 \subset N$ such that for $\epsilon \le \epsilon_0$, $\bar{\Gamma}^0_\epsilon = N_1$. In particular,

$$\bar{\Gamma}_\epsilon \supset N_1. \tag{11.44}$$

From the definition it is immediate that $\bar{\Gamma}_{\epsilon_0}$ is a closed subgroup of N. By Cartan's theorem (see e.g. [Kn, §0.4]), any closed subgroup of a Lie group is a closed Lie subgroup, and in particular, $\bar{\Gamma}_{\epsilon_0}$ and $N_1 = \bar{\Gamma}^0_{\epsilon_0}$ are closed submanifolds of N. Therefore, there exists $\epsilon_1 < \epsilon_0$ such that

$$B(e, \epsilon_1) \cap \bar{\Gamma}_{\epsilon_0} = B(e, \epsilon_1) \cap \bar{\Gamma}^0_{\epsilon_0} = B(e, \epsilon_1) \cap N_1.$$

Then, for $\epsilon < \epsilon_1 < \epsilon_0$,

$$\Gamma_\epsilon \cap B(e, \epsilon_1) \subset \bar{\Gamma}_{\epsilon_0} \cap B(e, \epsilon_1) \subset N_1.$$

Therefore, $\Gamma_\epsilon \subset N_1$, and hence $\bar{\Gamma}_\epsilon \subset N_1$. In view of (11.44), the lemma follows. $\square$

Lemma 11.13. *Suppose N is a nilpotent Lie group, and let $S \subset N$ be an (infinite) subset. For each $\epsilon > 0$ let $\Gamma_\epsilon \subset N$ denote the subgroup of N generated by the elements $\gamma \in S \cap B(e, \epsilon)$. Suppose that for all $\epsilon > 0$, Γ_ϵ is dense in N.*

Then, for every $\epsilon > 0$ there exist $0 < \theta < \epsilon$ (depending on ϵ and S) such that for every $\gamma \in \Gamma_\theta$ with $d(\gamma, e) < \theta$ there exists $n \in \mathbb{N}$ and for $1 \le i \le n$ elements $\gamma_i \in S$ with

$$\gamma = \gamma_n \dots \gamma_1 \tag{11.45}$$

and for each $1 \le j \le n$,

$$d(\gamma_j \dots \gamma_1, e) < \epsilon. \tag{11.46}$$

Proof. We will proceed by induction on $\dim N$. Let $N' = [N, N]$ and let $S' = [S \cap B(e, \epsilon/4), S \cap B(e, \epsilon/4)] \subset N'$. For $\delta > 0$ let Γ'_δ denote the subgroup of N' generated by $S' \cap B(e, \delta)$. Since (for sufficiently small δ) $[B(e, \delta), B(e, \delta)] \subset B(e, \delta)$, we have, for $\delta < \epsilon/4$,

$$\overline{\Gamma'_\delta} \supset \overline{[\Gamma_\delta, \Gamma_\delta]} = [\bar{\Gamma}_\delta, \bar{\Gamma}_\delta] = [N, N] = N'.$$

Therefore, $S' \subset N'$ satisfies the conditions of the Lemma. Let $\epsilon' > 0$ be such that

$$B(e, \epsilon') B(e, \epsilon') \subset B(e, \epsilon/100). \tag{11.47}$$

Since $\dim N' < \dim N$, by the inductive assumption there exist $0 < \theta' < \epsilon'$ such that for any $\gamma' \in \Gamma'_{\theta'}$ with $d(\gamma', e) < \theta'$, there exist $\gamma'_i \in S'$ such that (11.45) holds, and (11.46) holds with ϵ' in place of ϵ.

Suppose $\epsilon > \eta > 0$. By construction, N/N' is abelian. Then, since $\bar{\Gamma}_\eta = N$, there exists a finite set

$$S_0 \equiv \{\lambda_1, \dots, \lambda_k\} \subset \Gamma_\eta \cap S$$

so that $\lambda_1 N', \dots, \lambda_k N'$ form a basis over $\mathbb{R}$ for the vector space N/N'. Let Λ denote the subgroup generated by the λ_i, and let $F' \subset N/N'$ denote the parallelogram centered at the origin whose sides are parallel to the vectors $\lambda_i N'$. Then F' is a fundamental domain for the action of Λ on N/N', and

$$\operatorname{diam} F' = O(\eta). \tag{11.48}$$

Let N_0 be a complement to N' in N. We can choose N_0 to be a smooth manifold transversal to N' (N_0 need not be a subgroup). Let $\pi : N \to N/N'$ be the natural map, and let $\pi^{-1} : N/N' \to N_0$ be the inverse. Let $F = \pi^{-1}(F')$. We can now choose η sufficiently small so that $F \subset B(e,\rho)$, where $\theta' > \rho > \eta > 0$ is such that

$$B(e,\rho)^5 \cap N' = [B(e,\rho)B(e,\rho)B(e,\rho)B(e,\rho)B(e,\rho)] \cap N' \subset B(e,\theta') \cap N'.$$

We now choose $\theta > 0$ so that $B(e,\theta) \subset F\mathcal{O}$ where $\mathcal{O} \subset N' \cap B(e,\rho)$ is some neighborhood of the origin. We now claim that for any $x \in F\mathcal{O}$ and any $s \in B(e,\theta)$, there exist $\lambda \in S_0 \cup S_0^{-1}$ and $\gamma' \in \Gamma'_{\theta'}$ such that $\gamma'\lambda' x \in F\mathcal{O}$. Indeed, since $B(e,\theta)N' \subset FN'$, for any $x \in FN'$,

$$B(x,\theta)N' \subset \bigcup_{\lambda \in S_0 \cup S_0^{-1}} \lambda B(x,\theta)N'.$$

Thus, we can find $\lambda' \in S_0 \cup S_0^{-1}$ such that $\lambda' x \in FN'$. Since $\Gamma'_{\theta'}$ is dense in N', there exists $\gamma' \in \Gamma'_{\theta'}$ such that $\gamma'\lambda' x \in F\mathcal{O}$, completing the proof of the claim.

Now suppose $\gamma \in \Gamma_\theta$ and $\gamma \in B(e,\theta) \subset F\mathcal{O}$. Then, we have

$$\gamma = s_n \dots s_1, \text{ where } s_i \in S \cap B(e,\theta). \tag{11.49}$$

Note that $s_1 \in F\mathcal{O}$. We now define elements $\lambda'_j \in S_0 \cup S_0^{-1}$ and $\gamma'_j \in \Gamma'_{\theta'}$ inductively as follows. At every stage of the induction, we will have $x_j \equiv \gamma'_j s_j \dots \lambda'_1\gamma'_1 s_1 \in F\mathcal{O}$. Suppose $\gamma'_1, \dots, \gamma'_{j-1}$ and $\lambda'_1, \dots \lambda'_{j-1}$ have already been chosen. Now choose $\lambda'_j \in S_0 \cup S_0^{-1}$ and $\gamma'_j \in \Gamma'_{\theta'}$ so that $x_j = \gamma'_j\lambda'_j x_{j-1} \in F\mathcal{O}$. Such λ'_j and γ'_j exist by the claim.

Note that

$$\gamma'_j = x_j x_{j-1}^{-1}(\lambda'_j)^{-1} \in (F\mathcal{O})(F\mathcal{O})^{-1}(S_0 \cup S_0^{-1}) \subset B(e,\rho)^5 \subset B(e,\theta').$$

Since $x_n = \lambda'_n\gamma'_n s_n \dots \lambda'_1\gamma'_1 s_1 \in FN'$, we have $\lambda'_n s_n \dots \lambda'_1 s_1 \in FN'$. Also $\gamma = s_n \dots s_1 \in B(x,\theta) \subset FN'$. Since FN' is a fundamental domain for the action of Λ on N/N', $\lambda'_n \dots \lambda'_1 \in N'$. Thus,

$$\gamma = \gamma'\lambda'_n\gamma'_n s_n \dots \lambda'_1\gamma'_1 s_1, \tag{11.50}$$

where $\gamma' \in N'$. We have

$$\gamma' = \gamma x_n^{-1} \in B(e,\theta)(F\mathcal{O})^{-1} \subset B(e,\theta').$$

For notational convenience, denote γ' by γ'_{n+1}. By the inductive assumption, for $1 \le i \le n+1$, we can express $\gamma'_i = s'_{i1} \dots s'_{in_i}$ such that $s'_{ij} \in S' \cap B(e,\theta')$ and so that for all i, j,

$$d(s'_{ij} \dots s'_{i1}, e) \le \epsilon'.$$

We now substitute this into (11.50). Finally, we express each s'_{ij} as a commutator of elements of $S \cap B(e,\epsilon/4)$. Then, in view of (11.47), the resulting word satisfies (11.46). □

Proposition 11.14. *Suppose N is a nilpotent Lie group, $\mathcal{O}$ a neighborhood of the identity in N, and μ a measure on N supported on $\mathcal{O}$. Suppose $S \subset N$ is a subset containing elements arbitrarily close to (and distinct from) e, and suppose for each $\gamma \in S$,*

$$\gamma_* \mu \propto \mu \tag{11.51}$$

on $\mathcal{O} \cap \gamma^{-1}\mathcal{O}$ where both sides make sense. Then, there exists a connected subgroup H of N, such that for all $h \in H$, $h_*\mu \propto \mu$. Furthermore, if U is a connected subgroup of N and S contains arbitrarily small elements not contained in U, then H is not contained in U.

Proof. Let N_1 and ϵ_1 be as in Lemma 11.12. By our assumptions on S, N_1 is non-trivial (and also N_1 is not contained in U). Now suppose $\epsilon > 0$ is such that $B(e, \epsilon) \subset \mathcal{O}$, and let $\theta > 0$ be as in Lemma 11.13, with N replaced by N_1. Without loss of generality, we may assume that $\theta < \epsilon_1$. Let Γ_θ be the subgroup of N_1 generated by $S \cap B(e, \theta)$. Since $\theta < \epsilon_1$, Γ_θ is dense in N_1. Now suppose $\bar{\gamma} \in N_1$, and $d(\bar{\gamma}, e) < \theta$. Then, there exists $\gamma_k \in \Gamma_\theta$ such that $\gamma_k \to \gamma$, and $d(\gamma_k, e) < \theta$. We can write each $\gamma_k = \gamma_{k,n} \dots \gamma_{k,1}$ as in Lemma 11.13. Then, by applying (11.51) repeatedly, we get that $(\gamma_k)_* \mu \propto \mu$. Then, taking the limit as $k \to \infty$ we see that $(\bar{\gamma})_* \mu \propto \mu$. Thus, μ is invariant (up to normalization) under a neighborhood of the origin in N_1. $\square$

12. The Inductive Step

Proposition 12.1. *Suppose ν is a P-invariant measure on X. Let the subspace $\mathcal{L}^-(x) \subset W^-(x)$ be the smallest such that for almost all x, the conditional measure $\nu_{W^-[x]}$ is supported on $\mathcal{L}^-[x]$. Write $\mathcal{L}^-(x) = \pi_x^-(\mathcal{L}(x))$, where $\mathcal{L}(x) \subset W(x)$ (see (2.2) and (2.3)).*

Suppose $U^+(x)$ is an system of subgroups of $\mathcal{G}_+(x)$ containing $P \subset SL(2, \mathbb{R})$ such that for almost all x, the conditional measures $\nu_{U^+[x]}$ are induced from the Haar measure on $U^+(x)$. Write $U^+(x) = \pi_x^+ \circ U(x) \circ (\pi_x^+)^{-1}$, where $U[x] \subset W[x]$, see (2.3). Suppose $\mathcal{L}(x) \not\subset U(x)$, where we identify the subspace $\mathcal{L}(x)$ with the group of translations by vectors in the subspace. Then, there exists an equivariant system of subgroups $U^+_{new}(x)$ of $\mathcal{G}_+(x)$ such that for almost all x, $U^+_{new}(x)$ strictly contains $U^+(x)$ and the conditional measures $\nu_{U^+_{new}[x]}$ are induced from the Haar measure on $U^+_{new}(x)$.

The rest of §12 will consist of the proof of Proposition 12.1. We assume that $\mathcal{L}(x)$ and $U(x)$ are as in Proposition 12.1, and that $\mathcal{L}(x) \not\subset U(x)$. The argument has been outlined in §2.3, and we have kept the same notation (in particular, see Figure 1).

Let $f_{ij}(x)$ be the measures on $W^+(x)$ introduced in §11. Let $P^+(x, y)$ be the map introduced in §4.2. Proposition 12.1 will be derived from the following:

Proposition 12.2. *Suppose U, $\mathcal{L}$ are as in Proposition 12.1, and $\mathcal{L} \not\subset U$. Then there exists $0 < \delta_0 < 0.1$, a subset $K_* \subset X$ with $\nu(K_*) > 1 - \delta_0$ such that all the functions*

f_{ij}, $ij \in \tilde{\Lambda}$ are uniformly continuous on K_*, and $C > 1$ such that for every $\epsilon > 0$ there exists a subset $E \subset K_*$ with $\nu(E) > \delta_0$, such that for every $x \in E$ there exists $ij \in \tilde{\Lambda}$ and $y \in \mathcal{C}_{ij}[x] \cap K_*$ with

$$C^{-1}\epsilon \leq hd_x(U^+[x], U^+[y]) \leq C\epsilon \tag{12.1}$$

and (on the domain where both are defined)

$$f_{ij}(y) \propto P^+(x,y)_* f_{ij}(x). \tag{12.2}$$

We now begin the proof of Proposition 12.2.

Choice of parameters #1. Let $\epsilon > 0$ be arbitrary and $\eta > 0$ be arbitrary. Fix $\theta > 0$ as in Proposition 10.1 and Proposition 10.2. We then choose $\delta > 0$ sufficiently small; the exact value of δ will we chosen at the end of this section. All subsequent constants will depend on δ. (In particular, $\delta \ll \theta$; we will make this more precise below).

We will show that Proposition 12.2 holds with $\delta_0 = \delta/10$. Let $K_* \subset X$ be any subset with $\nu(K_*) > 1 - \delta_0$ on which all the functions f_{ij} are uniformly continuous. It is enough to show that there exists $C = C(\delta)$ such that for any $\epsilon > 0$ and for an arbitrary compact set $K_{00} \subset X$ with $K_{00} \geq (1 - 2\delta_0)$, there exists $x \in K_{00} \cap K_*$ and $y \in \mathcal{C}_{ij}[x] \cap K_*$ satisfying (12.1) and (12.2). Thus, let $K_{00} \subset X$ be an arbitrary compact set with $\nu(K_{00}) > 1 - 2\delta_0$.

We can choose a compact set $K_0 \subset K_{00} \cap K_*$ with $\nu(K_0) > 1 - 5\delta_0 = 1 - \delta_0/2$ so that Proposition 11.4 holds.

Let $\kappa > 1$ and $\ell_0 > 0$ be as in Proposition 7.4. Without loss of generality, assume $\delta < 0.01$. We now choose a subset $K \subset X$ with $\nu(K) > 1 - \delta$ such that the following hold:

- There exists a number $T_0(\delta)$ such that for any $x \in K$ and any $T > T_0(\delta)$,
$$\{t \in [-T/2, T/2] \;:\; g_t x \in K_0\} \geq 0.9T.$$
(This can be done by the Birkhoff ergodic theorem).
- Proposition 8.5 (a) holds.
- Proposition 10.1 holds.
- There exists a constant $C = C(\delta)$ such that for $x \in K$, $C_3(x)^2 < C(\delta)$ where C_3 is as in Proposition 10.2.
- Lemma 4.12 holds for $K = K(\delta)$ and $C_1 = C_1(\delta)$.
- There exists a constant $C' = C'(\delta)$ such that for $x \in K$, $C_1(x) < C'$, $C_2(x) < C'$ and $C(x) < C'$ where $C_1(x)$, $C_2(x)$ and $C(x)$ are as in Lemma 6.7.
- Lemma 6.8 holds for K.

Let
$$\tilde{\mathcal{D}}_{00}(q_1) = \tilde{\mathcal{D}}_{00}(q_1, K_{00}, \delta, \epsilon, \eta) = \{t > 0 \;:\; g_t q_1 \in K\}.$$
For $ij \in \tilde{\Lambda}$, let
$$\tilde{\mathcal{D}}_{ij}(q_1) = \tilde{\mathcal{D}}_{ij}(q_1, K_{00}, \delta, \epsilon, \eta) = \{\hat{\tau}_{ij}(q_1, t) \;:\; g_t q_1 \in K, \quad t > 0\}.$$

Then by the ergodic theorem and (9.5), there exists a set $K_{\mathcal{D}} = K_{\mathcal{D}}(K_{00}, \delta, \epsilon, \eta)$ with $\nu(K_{\mathcal{D}}) \geq 1 - \delta$ and $\ell_{\mathcal{D}} = \ell_{\mathcal{D}}(K_{00}, \delta, \epsilon, \eta) > 0$ such that that for $q_1 \in K_{\mathcal{D}}$ and all $ij \in \{00\} \cup \tilde{\Lambda}$, $\tilde{\mathcal{D}}_{ij}(q_1)$ has density at least $1 - 2\kappa\delta$ for $\ell > \ell_{\mathcal{D}}$. Let

$$E_2(q_1, u) = E_2(q_1, u, K_{00}, \delta, \epsilon, \eta) = \{\ell \ : \ g_{\hat{\tau}_{(\epsilon)}(q_1,u,\ell)} u q_1 \in K\},$$

$$E_3(q_1, u) = E_3(q_1, u, K_{00}, \delta, \epsilon, \eta) = \\ = \{\ell \in E_2(q_1, u) \ : \ \forall ij \in \tilde{\Lambda}, \quad \hat{\tau}_{ij}(uq_1, \hat{\tau}_{(\epsilon)}(q_1, u, \ell)) \in \tilde{\mathcal{D}}_{ij}(q_1)\}.$$

Claim 12.3. *There exists $\ell_3 = \ell_3(K_{00}, \delta, \epsilon, \eta) > 0$, a set $K_3 = K_3(K_{00}, \delta, \epsilon, \eta)$ of measure at least $1 - c_3(\delta)$ and for each $q_1 \in K_3$ a subset $Q_3 = Q_3(q_1, K_{00}, \ell, \delta, \epsilon, \eta) \subset \mathcal{B}$ of measure at least $(1 - c_3'(\delta))|\mathcal{B}|$ such that for all $q_1 \in K_3$ and $u \in Q_3$, the density of $E_3(q_1, u)$ (after length ℓ_3) is at least $1 - c_3''(\delta)$, and we have $c_3(\delta)$, $c_3'(\delta)$ and $c_3''(\delta) \to 0$ as $\delta \to 0$.*

Proof of claim. We choose $K_3 = K_1 \cap K_{\mathcal{D}}$. Suppose $q_1 \in K_3$, and $u \in Q_1(\delta, \epsilon)$, and $uq_1 \in K_3$. Let

$$E_{bad} = \{t \ : \ g_t u q_1 \in K^c\}.$$

Then, since $uq_1 \in K_{\mathcal{D}}$, the density of E_{bad} is at most $2\kappa\delta$. We have

$$E_2(q_1, u)^c = \{\ell \ : \ \hat{\tau}_{(\epsilon)}(q_1, u, \ell) \in E_{bad}\}.$$

Then, by Proposition 7.4, the density of $E_2(q_1, u)$ is at least $1 - 4\kappa^2\delta$.

Let

$$\hat{\mathcal{D}}(q_1, u) = \hat{\mathcal{D}}(q_1, u, K_{00}, \delta, \epsilon, \eta) = \{t \ : \ \forall ij \in \tilde{\Lambda}, \quad \hat{\tau}_{ij}(uq_1, t) \in \tilde{\mathcal{D}}_{ij}(q_1)\}.$$

Since $q_1 \in K_{\mathcal{D}}$, for each j, the density of $\tilde{\mathcal{D}}_{ij}(q_1)$ is at least $1 - 2\kappa\delta$. Then, by (9.5), the density of $\hat{\mathcal{D}}(q_1, u)$ is at least $(1 - 4|\tilde{\Lambda}|\kappa^2\delta)$. Now

$$E_3(q_1, u) = E_2(q_1, u) \cap \{\ell \ : \ \hat{\tau}_{(\epsilon)}(q_1, u, \ell) \in \hat{\mathcal{D}}(q_1, u)\}.$$

Now the claim follows from Proposition 7.4. $\square$

Claim 12.4. *There exists set $\mathcal{D}_4 = \mathcal{D}_4(K_{00}, \delta, \epsilon, \eta) \subset \mathbb{R}^+$, a number $\ell_4 = \ell_4(K_{00}, \delta, \epsilon, \eta) > 0$ so that $\mathcal{D}_4$ has density at least $1 - c_4(\delta)$ after length ℓ_4, and for $\ell \in \mathcal{D}_4$ a subset $K_4(\ell) = K_4(\ell, K_{00}, \delta, \epsilon, \eta) \subset X$ with $\nu(K_4(\ell)) > 1 - c_4'(\delta)$, such that for any $q_1 \in K_4(\ell)$ there exists a subset $Q_4(q_1, \ell) \subset \mathcal{B}$ with density at least $1 - c_4''(\delta)$, so that for all $\ell \in \mathcal{D}_4(\ell)$, for all $q_1 \in K_4(\ell)$ and all $u \in Q_4(q_1, \ell)$,*

$$\ell \in E_3(q_1, u) \subset E_2(q_1, u). \tag{12.3}$$

(We have $c_4(\delta)$, $c_4'(\delta)$ and $c_4''(\delta) \to 0$ as $\delta \to 0$).

Proof of Claim. This follows from applying Fubini's theorem to $X \times \mathcal{B} \times \mathbb{R}$. $\square$

Suppose $\ell \in \mathcal{D}_4$. We now apply Lemma 5.2 with $K' = g_{-\ell} K_4(\ell)$ and $\epsilon_1 = c_5'(\delta)$ as in the last statement of Lemma 5.2. We denote the resulting set K by $K_5(\ell) = K_5(\ell, K_{00}, \delta, \epsilon, \eta)$. In view of the choice of ϵ_1, we have $\nu(K_5(\ell)) \geq 1 - c_5(\delta)$, where $c_5(\delta) \to 0$ and $c_5'(\delta) \to 0$ as $\delta \to 0$.

Let $\mathcal{D}_5 = \mathcal{D}_4$ and let $K_6(\ell) = g_\ell K_5(\ell)$.

Choice of parameters #2: Choice of q, q', q_1' (depending on δ, ϵ, q_1, ℓ). Suppose $\ell \in \mathcal{D}_5$ and $q_1 \in K_6(\ell)$. Let $q = g_{-\ell} q_1$. Then, $q \in K_5(\ell)$. Let $\mathcal{A}(q, u, \ell, t)$ be as in §6, and for $u \in Q_4(q_1)$ let $\mathcal{M}_u$ be the subspace of Lemma 5.1 applied to the linear map $\mathcal{A}(q_1, u, \ell, \hat{\tau}_{(\epsilon)}(q_1, u, \ell))$. By Lemma 5.2 and the definition of $K_5(\ell)$, we can choose $q' \in \mathcal{L}^-[q] \cap g_{-\ell} K_4$ with $\rho'(\delta) \leq d(q, q') \leq 1$ and so that (5.8) holds with $\epsilon_1(\delta) = c_5'(\delta)$. Let $q_1' = g_\ell q_1$. Then $q_1' \in K_4$.

Standing Assumption. We assume $\ell \in \mathcal{D}_5$, $q_1 \in K_6(\ell)$ and q, q', q_1' are as in Choice of parameters #2.

The maps ψ, ψ', ψ''. For $u \in \mathcal{B}$, let

$$\psi(u) = g_{\hat{\tau}_{(\epsilon)}(q_1, u, \ell)} u q_1, \qquad \psi'(u) = g_{\hat{\tau}_{(\epsilon)}(q_1, u, \ell)} u q_1' \qquad \psi''(u) = g_{\hat{\tau}_{(\epsilon)}(q_1', u, \ell)} u q_1'.$$

Claim 12.5. *We have*

$$\psi(Q_4(q_1, \ell)) \subset K, \quad \textit{and} \quad \psi''(Q_4(q_1', \ell)) \subset K. \tag{12.4}$$

Proof of Claim. Suppose $u \in Q_4(q_1, \ell)$. Since $q_1 \in K_4$ and $\ell \in \mathcal{D}_4$, it follows from (12.3) that $\ell \in E_2(q_1, u)$, and then from the definition of $E_2(q_1, u)$ is follows that $\psi(u) \in K$. Hence $\psi(Q_4(q_1, \ell)) \subset K$. Similarly, since $q_1' \in K_4$, $\psi''(Q_4(q_1', \ell)) \subset K$, proving (12.4). $\square$

The numbers t_{ij} and t_{ij}'. Suppose $u \in Q_4(q_1, \ell)$, and suppose $ij \in \tilde{\Lambda}$. Let t_{ij} be defined by the equation

$$\hat{\tau}_{ij}(u q_1, \hat{\tau}_{(\epsilon)}(q_1, u, \ell)) = \hat{\tau}_{ij}(q_1, t_{ij}), \tag{12.5}$$

Then, since $\ell \in \mathcal{D}_4$ and in view of (12.3), we have $\ell \in E_3(q_1, u)$. In view of the definition of E_3, it follows that

$$g_{t_{ij}} q_1 \in K. \tag{12.6}$$

Similarly, suppose $u \in Q_4(q_1', \ell)$ and $ij \in \tilde{\Lambda}$. Let t_{ij}' be defined by the equation

$$\hat{\tau}_{ij}(u q_1', \hat{\tau}_{(\epsilon)}(q_1', u, \ell)) = \hat{\tau}_{ij}(q_1, t_{ij}'). \tag{12.7}$$

Then, by the same argument,

$$g_{t_{ij}'} q_1' \in K. \tag{12.8}$$

The map $\mathbf{v}(u)$ and the generalized subspace $\mathcal{U}(u)$. For $u \in \mathcal{B}$, let

$$\mathbf{v}(u) = \mathcal{A}(q, u, \ell, t)(F(q) - F(q'))$$

where $t = \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$, F is as in §5 and $\mathcal{A}(\cdot, \cdot, \cdot, \cdot)$ is as in §6.1. Then, if $\mathbf{v}(u) = \mathbf{j}(M'', v'')$ let $\mathcal{U}(u) \equiv \mathcal{U}_{\psi(u)}(M'', v'')$ denote the generalized affine subspace corresponding to $\mathbf{v}(u)$. Thus, $\mathcal{U}(u)$ is the approximation to $U^+[\psi'(u')]$ defined in Lemma 6.7.

Claim 12.6. *There exists a subset $Q_5 = Q_5(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset \mathcal{B}$ with $|Q_5| \geq (1 - c_5''(\delta))|\mathcal{B}|$ (with $c_5''(\delta) \to 0$ as $\delta \to 0$), and a number $\ell_5 = \ell_5(\delta, \epsilon)$ such that if $\ell > \ell_5$, for all $u \in Q_5$, there exists $u' \in \mathcal{B}$ such that*

$$C_1(\delta)\epsilon \leq hd_{\psi(u)}(U^+[\psi(u)], U^+[\psi'(u')]) \leq C_2(\delta)\epsilon, \tag{12.9}$$

$$hd_{\psi(u)}(U^+[\psi'(u')], \mathcal{U}(u)) \leq C_7(\delta)e^{-\alpha\ell}, \tag{12.10}$$

where α depends only on the Lyapunov spectrum. Also,

$$C_1'(\delta)\epsilon \leq \|\mathbf{v}(u)\| \leq C_2'(\delta)\epsilon, \tag{12.11}$$

Proof of claim. Let $Q_5 \subset Q_4$ be such that for all $u \in Q_5$, $d(u, \partial\mathcal{B}) > \delta$, and

$$d(F(q) - F(q'), \mathcal{M}_u) \geq \beta(\delta) \tag{12.12}$$

where F is as in §5, ans $\mathcal{M}_u$ be the subspace of Lemma 5.1 applied to the linear map $\mathcal{A}(q_1, u, \ell, \hat{\tau}_{(\epsilon)}(q_1, u, \ell))$, where $\mathcal{A}(,,,)$ is as in §6. Let Q_5' denote the set of $u \in Q_4$ such that (12.12) holds for u. Then, by the choice of q',

$$|Q_5'| \geq |Q_4| - c_5'(\delta) \geq 1 - c_5'(\delta) - c_4'(\delta).$$

Then, $Q_5 = \{u \in Q_5' \ : \ d(u, \partial\mathcal{B}) > \delta\}$, hence $|Q_5| \geq (1 - c_5'(\delta) - c_4'(\delta)) - c_n\delta$, where c_n depends only on the dimension.

Suppose $u \in Q_5$. Then, by Lemma 6.8, we have $d(u, u') = O_\delta(e^{-\alpha\ell})$. Then, assuming ℓ is sufficiently large (depending on δ) we have $u' \in \mathcal{B}$.

We have $C(\delta)^{-1}\epsilon \leq \|\mathcal{A}(q_1, u, \ell, t)\| \leq C(\delta)\epsilon$ by the definition of $t = \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$. We now apply Lemma 5.1 to the linear map $\mathcal{A}(q_1, u, \ell, t)$. Then, for all $u \in Q_5$,

$$c(\delta)\|\mathcal{A}(q_1, u, \ell, t)\| \leq \|\mathcal{A}(q_1, u, \ell, t)(F(q) - F(q'))\| \leq \|\mathcal{A}(q_1, u, \ell, t)\|.$$

Therefore,

$$C'(\delta)^{-1}\epsilon \leq \|\mathcal{A}(q_1, u, \ell, t)(F(q) - F(q'))\| \leq C'(\delta)\epsilon$$

This immediately implies (12.11), in view of the definition of $\mathbf{v}(u)$. We now apply Lemma 6.7. (We assume ϵ is sufficiently small so that (6.18) holds). Now (12.9) follows from (6.21). Also (12.10) follows from (6.19). $\square$

Claim 12.7. *There exists a subset $Q_6(q_1, \ell) = Q_6(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset \mathcal{B}$ with $|Q_6(q_1, \ell)| > (1 - c_6'(\delta))|\mathcal{B}|$ and with $c_6'(\delta) \to 0$ as $\delta \to 0$ such that for all $u \in Q_6(q_1, \ell)$ there exists $u' \in Q_4(q_1', \ell)$ such that (12.9) and (12.10) hold.*

Proof of Claim. We decompose $\mathcal{B} = \bigsqcup_{j=1}^N P_j$, $\mathcal{B} = \bigsqcup_{j=1}^N P'_j$ so that for $u \in P_j$, $u' \in P'_j$, and (12.9) and (12.10) hold. By Lemma 6.8, $\kappa^{-1}|P_j| \leq |P'_j| \leq \kappa|P_j|$, where κ depends only on the Lyapunov spectrum. Then, applying Lemma 7.5 with $Q = Q_5(q_1, \ell)$, $Q' = Q_4(q'_1, \ell)$, the claim follows. $\square$

Claim 12.8. *There exists a constants $c_7(\delta) > 0$ and $c'_7(\delta)$ with $c_7(\delta) \to 0$ and $c'_7(\delta) \to 0$ as $\delta \to 0$ and a subset $K_7 = K_7(K_{00}, \delta, \epsilon, \eta)$ with $\nu(K_7) > 1 - c_7(\delta)$ such that for $q_1 \in K_7(\delta)$,*

$$|\mathcal{B}(q_1) \cap Q_6(q_1, \ell)| \geq (1 - c'_7(\delta))|\mathcal{B}(q_1)|.$$

Proof of Claim. Given $\delta > 0$, there exists $c''_7(\delta) > 0$ with $c''_7(\delta) \to 0$ as $\delta \to 0$ and a compact set $K'_7 \subset X$ with $\nu(K'_7) > 1 - c''_7(\delta)$, such that for $q_1 \in K'_7$, $|\mathcal{B}(q_1)| \geq c'_6(\delta)^{1/2}|\mathcal{B}|$. Then, for $q_1 \in K'_7 \cap K_6$,

$$|\mathcal{B}(q_1) \cap Q_6(q_1, \ell)^c| \leq |Q_6(q_1, \ell)^c| \leq c'_6(\delta)|\mathcal{B}| \leq c'_6(\delta)^{1/2}|\mathcal{B}(q_1)|.$$

Thus, the claim holds with $c_7(\delta) = c_6(\delta) + c''_7(\delta)$ and $c'_7(\delta) = c'_6(\delta)^{1/2}$. $\square$

Standing Assumption. We assume that $q_1 \in K_7$.

Claim 12.9. *There exists a subset $Q^*_7(q_1, \ell) = Q^*_7(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_7(q_1, \ell)$ with $|Q^*_7| \geq (1 - c^*_7(\delta))|\mathcal{B}(q_1)|$ such that for $u \in Q^*_7$ and any $t > \ell_7(\delta)$ we have*

$$|\mathcal{B}_t(uq_1) \cap Q_7(q_1, \ell)| \geq (1 - c^*_7(\delta))|\mathcal{B}_t(uq_1)|,$$

where $c^*_7(\delta) \to 0$ as $\delta \to 0$.

Proof. This follows immediately from Lemma 3.11. $\square$

Claim 12.10. *There exist a number $\ell_8 = \ell_8(K_{00}, \delta, \epsilon, \eta)$ and a constant $c_8(\delta)$ with $c_8(\delta) \to 0$ as $\delta \to 0$ and for every $\ell > \ell_8$ a subset $Q_8(q_1, \ell) = Q_8(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset \mathcal{B}[q_1]$ with $|Q_8(q_1, \ell)| \geq (1 - c_8(\delta))|\mathcal{B}[q_1]|$ so that for $u \in Q_8(q_1, \ell)$ we have*

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}(\psi(u))\right) \leq C_8(\delta)e^{-\alpha'\ell}, \tag{12.13}$$

where α' depends only on the Lyapunov spectrum.

Proof of claim. Let $L' > L_2(\delta)$ be a constant to be chosen later, where $L_2(\delta)$ is as in Proposition 8.5 (a). Also let $\ell_8 = \ell_8(\delta, \epsilon, K_{00}, \eta)$ be a constant to be chosen later. Suppose $\ell > \ell_8$, and suppose $u \in Q^*_7(q_1, \ell)$, so in particular $\psi(u) \in K$. Let $t \in [L', 2L']$ be such that Proposition 8.5 (a) holds for $\mathbf{v} = \mathbf{v}(u)$.

Let $B_u \subset \mathcal{B}(q_1)$ denote $\mathcal{B}_{\hat{\tau}_{(\epsilon)}(q_1, u, \ell) - t}(uq_1)$, (where $\mathcal{B}_t(x)$ is defined in §3). Suppose $u_1 \in B_u \cap Q_7(q_1, \ell)$, and write

$$\psi(u_1) = g_s u_2 g_t^{-1}\psi(u).$$

Then, $u_2 \in \mathcal{B}(g_t^{-1}\psi(u))$, $t \le 2L'$ and $s \le \kappa L'$, where κ depends only on the Lyapunov spectrum. Since $u \in Q_6(q_1,\ell)$ we know that there exists $u' \in \mathcal{B}$ such that (12.9) and (12.10) hold. Therefore, for some $u_1' \in \mathcal{B}$,

$$hd_{\psi(u_1)}((g_s u_2 g_t^{-1})\mathcal{U}(u), U^+[\psi'(u_1')]) = O(e^{\kappa' L'} e^{-\alpha\ell}),$$

where κ' and α depend only on the Lyapunov spectrum. Thus, using (12.10) at the point $\psi(u_1) \in K$ and using Lemma 6.6,

$$hd_{\psi(u_1)}((g_s u_2 g_t^{-1})\mathcal{U}(u), \mathcal{U}(u_1)) = O(e^{\kappa' L'} e^{-\alpha\ell}).$$

Therefore,

$$\|(g_s u_2 g_t^{-1})_* \mathbf{v}(u) - \mathbf{v}(u_1)\| = O(e^{\kappa' L'} e^{-\alpha\ell}). \tag{12.14}$$

In view of (12.11), $\|\mathbf{v}(u_1)\| \approx \epsilon$. Thus,

$$\left\| \frac{(g_s u_2 g_t^{-1})_* \mathbf{v}(u)}{\|(g_s u_2 g_t^{-1})_* \mathbf{v}(u)\|} - \frac{\mathbf{v}(u_1)}{\|\mathbf{v}(u_1)\|} \right\| = O_\epsilon(e^{\kappa' L' - \alpha\ell}).$$

But, by Proposition 8.5 (a), for $1-\delta$ fraction of $u_2 \in \mathcal{B}(g_t^{-1}\psi(u))$,

$$d\left(\frac{(g_t u_2 g_{-s})_* \mathbf{v}(u)}{\|(g_t u_2 g_{-s})_* \mathbf{v}(u)\|}, \mathbf{E}(\psi(u_1) \right) \le C(\delta) e^{-\alpha L'},$$

Note that

$$\mathcal{B}(g_t^{-1}\psi(u)) = g_{\hat{\tau}_{(\epsilon)}(q_1,u,\ell)-t} B_u.$$

Therefore, for $1-\delta$ fraction of $u_1 \in B_u$,

$$d\left(\frac{\mathbf{v}(u_1)}{\|\mathbf{v}(u_1)\|}, \mathbf{E}(\psi(u_1)) \right) \le C(\epsilon,\delta)[e^{\kappa' L' - \alpha\ell} + e^{-\alpha L'}] \tag{12.15}$$

We can now choose $L' > 0$ depending only on ℓ and the Lyapunov spectrum and $\ell_8 > 0$ so that for $\ell > \ell_8$ the right-hand-side of the above equation is at most $e^{-\alpha'\ell}$.

The collection of balls $\{B_u\}_{u \in Q_6^*(q_1,\ell)}$ are a cover of $Q_6^*(q_1,\ell)$. These balls satisfy the condition of Lemma 3.9 (b); hence we may choose a pairwise disjoint subcollection which still covers $Q_6^*(q_1,\ell)$. Then, by summing (12.15), we see that (12.13) holds for u in a subset $Q_8 \subset \mathcal{B}[q_1]$ of measure at least $(1-c_8(\delta))|\mathcal{B}[q_1]| = (1-\delta)(1-c_7^*(\delta))|\mathcal{B}[q_1]|$. $\square$

Claim 12.11. *There exists a subset $Q_8^*(q_1,\ell) = Q_8^*(q_1,\ell,K_{00},\delta,\epsilon,\eta) \subset Q_8(q_1,\ell)$ with $|Q_8^*| \ge (1-c_8^*(\delta))|\mathcal{B}(q_1)|$ such that for $u \in Q_8^*$ and any $t > \ell_8(\delta)$ we have*

$$|\mathcal{B}_t(uq_1) \cap Q_8(q_1,\ell)| \ge (1-c_8^*(\delta))|\mathcal{B}_t(uq_1)|,$$

where $c_8^*(\delta) \to 0$ as $\delta \to 0$.

Proof. This follows immediately from Lemma 3.11. □

Choice of parameters #3: Choice of δ. Let $\theta' = (\theta/2)^n$, where θ and n are as in Proposition 10.1. We can choose $\delta > 0$ so that

$$c_8^*(\delta) < \theta'/2. \tag{12.16}$$

Claim 12.12. *There exist sets $Q_9(q_1, \ell) = Q_9(q_1, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_8^*(q_1, \ell)$ with $|Q_9(q_1, \ell)| \geq (\theta'/2)(1 - \theta'/2)|\mathcal{B}(q_1)|$ and $\ell_9 = \ell_9(K_{00}, \delta, \epsilon, \eta)$, such that for $\ell > \ell_9$ and $u \in Q_9(q_1, \ell)$,*

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(\psi(u))\right) < 4\eta. \tag{12.17}$$

Proof of claim. Suppose $u \in Q_8^*(q_1, \ell)$. Then, by (12.13) we may write

$$\mathbf{v}(u) = \mathbf{v}'(u) + \mathbf{v}''(u),$$

where $\mathbf{v}'(u) \in \mathbf{E}(\psi(u))$ and $\|\mathbf{v}''(u)\| \leq C(\delta, \epsilon)e^{-\alpha'\ell}$. Then, by Proposition 10.1 applied with $L = L_0(\delta, \eta)$ and $\mathbf{v} = \mathbf{v}'(u)$, we get that for a at least θ'-fraction of $y \in \mathcal{F}_{\mathbf{v}'}[\psi(u), L]$,

$$d\left(\frac{R(\psi(u), y)\mathbf{v}'(u)}{\|R(\psi(u), y)\mathbf{v}'(u)\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < 2\eta.$$

Then, for at least θ'-fraction of $y \in \mathcal{F}_{\mathbf{v}'}[\psi(u), L]$, using Lemma 3.5,

$$d\left(\frac{R(\psi(u), y)\mathbf{v}(u)}{\|R(\psi(u), y)\mathbf{v}(u)\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < 3\eta + C(\epsilon, \delta)e^{2NL}e^{-\alpha'\ell}. \tag{12.18}$$

where N is as in Lemma 3.5. Let $B_u = \mathcal{B}_{\hat{\tau}_{(\epsilon)}(q_1, u, \ell) - L}(uq_1)$. In view of (12.14) and (12.11) there exists $C = C(\epsilon, \delta)$ such that

$$\mathcal{F}_{\mathbf{v}'}[\psi(u), L] \cap K \subset g_{[-C,C]}\psi(B_u) \quad \text{and} \quad \psi(B_u) \cap K \subset g_{[-C,C]}\mathcal{F}_{\mathbf{v}'}[\psi(u), L]$$

Then, by (12.18) and (12.16), for $(\theta'/2)$-fraction of $u_1 \in B_u$, $\psi(u_1) \in K$ and

$$d\left(\frac{R(\psi(u), \psi(u_1))\mathbf{v}(u)}{\|R(\psi(u), \psi(u_1))\mathbf{v}(u)\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(\psi(u_1))\right) < C_1(\epsilon, \delta)[3\eta + e^{2NL}e^{-\alpha'\ell}).]$$

Then, by (12.14), for $(\theta'/2)$-fraction of $u_1 \in B_u$,

$$d\left(\frac{\mathbf{v}(u_1)}{\|\mathbf{v}(u_1)\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(\psi(u_1))\right) < C_2(\epsilon, \delta)[3\eta + e^{2NL}e^{-\alpha'\ell} + e^{-\alpha'\ell}].$$

Hence, we may choose $\ell_9 = \ell_9(K_{00}, \epsilon, \delta, \eta)$ so that for $\ell > \ell_9$ the right-hand side of the above equation is at most 4η. Thus, (12.17) holds for $(\theta'/2)$-fraction of $u_1 \in B_u$.

The collection of balls $\{B_u\}_{u \in Q_8^*(q_1,\ell)}$ are a cover of $Q_8^*(q_1, \ell)$. These balls satisfy the condition of Lemma 3.9 (b); hence we may choose a pairwise disjoint subcollection which still covers $Q_8^*(q_1, \ell)$. Then, by summing over the disjoint subcollection, we see that the claim holds on a set E of measure at least $(\theta'/2)|Q_8^*| \geq (\theta'/2)(1 - c_8^*(\delta)) \geq (\theta'/2)(1 - \theta'/2)$. □

Choice of parameters #4: Choosing ℓ, q_1, q, q', q_1'. Choose $\ell > \ell_9(K_{00}, \epsilon, \delta, \eta)$. Now choose $q_1 \in K_7$, and let q, q', q_1' be as in Choice of Parameters #2.

Choice of parameters #5: Choosing u, u', q_2, q_2', $q_{3,ij}$, $q_{3,ij}'$ (depending on q_1, q_1', u, ℓ). Choose $u \in Q_9(q_1, \ell)$, $u' \in Q_4(q_1', \ell)$ so that (12.9) and (12.10) hold. We have $\psi(u) \subset K$ and $\psi''(u') \in K$. Note that in view of (12.9),

$$|\hat{\tau}_{(\epsilon)}(q_1, u', \ell) - \hat{\tau}_{(\epsilon)}(q_1', u', \ell)| = O(\delta),$$

therefore,

$$\psi'(u') \in g_{[-C,C]}K,$$

where $C = C(\delta)$.

By the definition of K we can find $C_4(\delta)$ and $s \in [0, C_4(\delta)]$ such that

$$q_2 \equiv g_s \psi(u) \in K_0, \qquad q_2' \equiv g_s \psi'(u') \in K_0. \tag{12.19}$$

Let $ij \in \tilde{\Lambda}$ be such that

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}_{[ij],bdd}(\psi(u))\right) \leq 4\eta \tag{12.20}$$

Also, in view of Lemma 6.7 and (9.5),

$$|t_{ij} - t_{ij}'| \leq C_5(\delta).$$

Therefore, by (12.6) and (12.8), we have

$$g_{t_{ij}} q_1 \in K, \quad \text{and} \quad g_{t_{ij}} q_1' \in g_{[-C_5(\delta), C_5(\delta)]}K.$$

By the definition of K, we can find $s'' \in [0, C_5''(\delta)]$ such that

$$q_{3,ij} \equiv g_{s''+t_{ij}} q_1 \in K_0, \quad \text{and} \quad q_{3,ij}' \equiv g_{s''+t_{ij}} q_1' \in K_0. \tag{12.21}$$

Let $\tau = s + \hat{\tau}_{(\epsilon)}(q_1, u, \ell)$, $\tau' = s'' + t_{ij}$. Then,

$$q_2 = g_\tau u q_1, \quad q_2' = g_\tau u' q_1', \quad q_{3,ij} = g_{\tau'} q_1, \quad q_{3,ij}' = g_{\tau'} q_1'.$$

We may write $q_2 = g_t^{ij} u q_1$, $q_{3,ij}' = g_{t'}^{ij} q_1$. Then, in view of (12.7),

$$|\tau - \tau'| \leq C_6(\delta).$$

Taking the limit as $\eta \to 0$. For fixed δ and ϵ, we now take a sequence of $\eta_k \to 0$ (this forces $\ell_k \to \infty$) and pass to limits along a subsequence. Let $\tilde{q}_2 \in K_0$ be the

limit of the q_2, and $\tilde{q}_2' \in K_0$ be the limit of the q_2'. We may also assume that along the subsequence $ij \in \tilde{\Lambda}$ is fixed, where ij is as in (12.20). We get

$$\frac{1}{C(\delta)}\epsilon \leq hd_{\tilde{q}_2}(U^+[\tilde{q}_2], U^+[\tilde{q}_2']) \leq C(\delta)\epsilon,$$

and in view of (12.20),

$$\tilde{q}_2' \in \mathcal{C}_{ij}(\tilde{q}_2).$$

Now, by Proposition 11.4 (c), we have

$$f_{ij}(\tilde{q}_2) \propto P^+(\tilde{q}_2, \tilde{q}_2')_* f_{ij}(\tilde{q}_2').$$

We have $\tilde{q}_2 \in K_0 \subset K_{00} \cap K_*$, and $\tilde{q}_2' \in K_0 \subset K_*$. This concludes the proof of Proposition 12.2. □

Applying the argument for a sequence of ϵ's tending to 0 . Take a sequence $\epsilon_n \to 0$. We now apply Proposition 12.2 with $\epsilon = \epsilon_n$. We get, for each n a set $E_n \subset K_*$ with $\nu(E_n) > \delta_0$ and with the property that for every $x \in E_n$ there exists $y \in \mathcal{C}_{ij}(x) \cap K_*$ such that (12.1) and (12.2) hold for $\epsilon = \epsilon_n$. Let

$$F = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \subset K_*,$$

(so F consists of the points which are in infinitely many E_n). Suppose $x \in F$. Then there exists a sequence $y_n \to x$ such that $y_n \in \mathcal{C}_{ij}[x]$, $y_n \notin U^+[x]$, and so that $f_{ij}(y_n) \propto P(x, y_n)_* f_{ij}(x)$. Then, (on the set where both are defined)

$$f_{ij}(x) \propto (\gamma_n)_* f_{ij}(x),$$

where $\gamma_n \in \mathcal{G}_+(x)$ is the affine map whose linear part is $P(x, y_n)$ and whose translational part is $y_n - x$. (Here we have used the fact that $y_n \in \mathcal{C}_{ij}[x]$, and thus by the definition of conditional measure, $f_{ij}(y_n) = (y_n - x)_* f_{ij}(x)$, where $(y_n - x)_* : W^+(x) \to W^+(x)$ is translation by $y_n - x$.)

Let B_Q denote the unit ball in the stabilizer $Q_+(x)$ of x in $\mathcal{G}_+(x)$. Let ν_Q denote the measure on $Q_+(x)$ which is the Haar measure on B_Q and zero outside of B_Q. Let $\mathcal{G}'(x)$ be a (analytic) complement to $Q_+(x)$ so that (locally) $\mathcal{G}_+(x) = \mathcal{G}'(x)Q_+(x)$. Let $\pi : \mathcal{G}_+(x) \to W^+[x] \times Q_+(x)$ be the map which takes $g'q \to (g'x, q)$ where $g' \in \mathcal{G}'(x)$, $q \in Q_+(x)$. Let $\tilde{f}_{ij}(x)$ denote the pullback of $f_{ij}(x) \times \nu_Q$ under π. Then, $\tilde{f}_{ij}(x)$ is a measure on $\mathcal{G}_+(x)$ which satisfies

$$(\gamma_n)_* \tilde{f}_{ij}(x) \propto \tilde{f}_{ij}(x)$$

on the set where both are defined. Then, by Proposition 11.14, for $x \in F$ there exists a connected subgroup $U^+_{new}(x)$ which strictly contains $U^+(x)$ such that for $u \in U^+_{new}(x)$, (on the domain where both are defined),

$$(u)_* \tilde{f}_{ij}(x) = e^{\beta(u,x)} \tilde{f}_{ij}(x), \tag{12.22}$$

where the normalization factor $\beta(u,x)$ is a measurable $\mathbb{R}$-valued cocycle. Since $\nu(F) > \delta_0 > 0$ and g_t is ergodic, for almost all $x \in F$ there exist arbitrarily large $t > 0$ so that $g_{-t}x \in F$. Then, (12.22) implies that $\beta(u,x) = 0$ almost everywhere (cf. [BQ, Proposition 7.4(b)]). Therefore, for almost all $x \in F$, the conditional measure of ν along the orbit $U^+_{new}[x]$ is the push-forward of the Haar measure on $U^+_{new}(x)$. Since $\nu(F) > \delta_0 > 0$, by the ergodicity of g_t, this statement holds for almost all x. This completes the proof of Proposition 12.1. □

13. Proof of Theorem 2.1.

Let $\mathcal{L}$ be as in §5 (or Proposition 12.1). Apply Proposition 12.1 to get an equivariant system of subgroups $U^+_{new}(x) \subset \mathcal{G}_+(x)$, such that for almost all x, the conditional measures $\nu_{U^+_{new}}(x)$ are induced from the Haar measure on $U^+_{new}(x)$. We now rename $U_{new}(x)$ to be $U(x)$. We may assume that the subspaces $U(x)$ are AN-equivariant, since otherwise we can enlarge them further. We identify $\mathcal{L}(x)$ with the subgroup of translations by elements of $\mathcal{L}(x)$. Then, we may consider $\mathcal{L}(x)$ to be a subgroup of the affine group $\mathcal{G}_+(x)$. If $\mathcal{L}(x) \not\subset U(x)$ we can (in view of Proposition 6.9) apply Proposition 12.1 again and repeat the process. When this process stops, the following hold:

- (a) $\mathcal{L}(x) \subset U_{trans}(x)$, where $U_{trans}(x)$ is the maximal pure-translation subgroup of $U(x)$.
- (b) The conditional measures $\nu_{U^+[x]}$ are induced from the Haar measure on $U^+[x]$. In particular, the conditional measures on the subspaces $U_{trans}[x]$ are Lebesgue.
- (c) The subgroups $U(x)$ and the subspaces $U_{trans}(x)$ are AN-equivariant. The subspaces $\mathcal{L}^-(x)$ are A-equivariant.
- (d) The conditional measures $\nu_{W^-[x]}$ are supported on $\mathcal{L}^-[x]$.

Let $H^1_\perp$ denote the subspace of $H^1(M,\Sigma,\mathbb{R})$ which is orthogonal to the $SL(2,\mathbb{R})$ orbit, see (2.1). Let I denote the Lyapunov exponents (with multiplicity) of the cocycle in $U_{trans} \cap H^1_\perp$, $J \subset I$ denote the Lyapunov exponents of the cocycle in $\mathcal{L} \cap H^1_\perp$. Since $U_{trans} \cap H^1_\perp$ is AN-invariant, by Theorem A.3 we have,

$$\sum_{i \in I} \lambda_i \geq 0. \tag{13.1}$$

Set $t = 1$. We now compute the entropy of g_t. We have, by Theorem B.7(i)

$$\frac{1}{t} h_\nu(g_t, W^+) \geq 2 + \sum_{i \in I}(1+\lambda_i) = 2 + |I| + \sum_{i \in I} \lambda_i \geq 2 + |I| \tag{13.2}$$

where the 2 comes from the direction of N, and for the last estimate we used (13.1). Also, by Theorem B.7(ii),

$$
\begin{aligned}
\frac{1}{t}h_\nu(g_{-t},W^-) &\le 2+\sum_{j\in J}(1-\lambda_j), && \text{where the 2 is the potential contribution of } \bar N\\
&\le 2+\sum_{i\in I}(1-\lambda_i) && \text{since } (1-\lambda_i)\ge 0 \text{ for all } i\\
&\le 2+|I| && \text{by (13.1)} \qquad (13.3)
\end{aligned}
$$

However, $h_\nu(g_t,W^+)=h_\nu(g_{-t},W^-)$. Therefore, all the inequalities in (13.2) and (13.3) are in fact equalities. In particular, $I=J$ (i.e. $\mathcal{L}=U_{trans}$), and

$$\frac{1}{t}h_\nu(g_{-t},W^-)=2+\sum_{i\in I}(1-\lambda_i).$$

By Theorem B.7(ii), this implies that the conditional measures $\nu_{\mathcal{L}^-}(x)$ are Lebesgue, and that ν is $\bar N$-invariant (where $\bar N$ is as in §1.1). Hence ν is $SL(2,\mathbb{R})$-invariant.

By the definition of $\mathcal{L}^-$, the conditional measures $\nu_{W^-[x]}$ are supported on $\mathcal{L}^-[x]$. Thus, the conditional measures $\nu_{W^-[x]}$ are (up to null sets) precisely the Lebesgue measures on $\mathcal{L}^-[x]$. Since ν is $SL(2,\mathbb{R})$-invariant, we can argue by symmetry that the conditional measures $\nu_{W^+[x]}$ are precisely the Lebesgue measures on the smallest subspace containing $U^+[x]$, and hence $U^+[x]=U_{trans}[x]$. Since $U_{trans}=\mathcal{L}$, this completes the proof of Theorem 2.1. $\square$

14. Random Walks

We choose a compactly supported absolutely continuous measure μ on $SL(2,\mathbb{R})$. We also assume that μ is spherically symmetric. Let ν be any ergodic μ-invariant stationary measure on X. By Furstenberg's theorem [NZ, Theorem 1.4]

$$\nu=\int_0^{2\pi}(r_\theta)_*\nu_0\,d\theta$$

where r_θ is as in §1.1 and ν_0 is a measure invariant under $P=AN\subset SL(2,\mathbb{R})$. Then, by Theorem 2.1, ν_0 is $SL(2,\mathbb{R})$-invariant. Therefore the stationary measure ν is also in fact $SL(2,\mathbb{R})$-invariant.

By Theorem 2.1, there is a $SL(2,\mathbb{R})$-equivariant family of subspaces $U(x)\subset H^1(M,\Sigma,\mathbb{R})$, and that the conditional measures of ν along $U_{\mathbb{C}}(x)$ are Lebesgue.

Lemma 14.1. *There exists a volume form $d\,\mathrm{Vol}(x)$ on $U(x)$ which is invariant under the $SL(2,\mathbb{R})$ action.*

Proof. The subspaces $p(U(x))$ form an invariant subbundle $p(U)$ of the Hodge bundle. By Theorem A.6 (a), (after passing to a finite cover) we may assume that

$p(U)$ is a direct sum of irreducible subbundles. Then, by Theorem A.6 (b) From we have an decomposition

$$p(U(x)) = U_{symp}(x) \oplus U_0(x)$$

where the symplectic form on U_{symp} is non-degenerate, the decomposition is orthogonal with respect to the Hodge inner product, and U_0 is isotropic. Then, by Theorem A.5 and Theorem A.4 the Hodge inner product on U_0 is equivariant under the $SL(2,\mathbb{R})$ action. Then we can define the volume form on $p(U)$ to be the product of the appropriate power of the symplectic form on U_{symp} and the volume form induced by the Hodge inner product on U_0. Since the cocycle acts trivially on $\ker p$, the normalized Lebesgue measure on $\ker p$ is well defined. Thus, the volume form on $p(U)$ naturally induces a volume form on U. □

Remark. In fact it follows from the results of [AEM] that U_0 is trivial.

Lemma 14.2. *There exists an $SL(2,\mathbb{R})$-equivariant subbundle $p(U)^\perp \subset H^1(M,\mathbb{R})$ such that*

$$p(U(x)) \oplus p(U(x))^\perp = H^1(M,\mathbb{R}). \tag{14.1}$$

Proof. This follows from the proof of Theorem A.6. □

The subbundles $\mathcal{L}_k$. By Theorem A.6 we have

$$p(U(x))^\perp = \bigoplus_{k \in \hat{\Lambda}} \mathcal{L}_k(x), \tag{14.2}$$

where for each k, $\mathcal{L}_k$ is an $SL(2,\mathbb{R})$-equivariant subbundle of the Hodge bundle. Note that $\mathcal{L}_k(x)$ is symplectically orthogonal to the $SL(2,\mathbb{R})$ orbit of x. In view of Theorem A.6 we may also assume that on any finite cover of X each $\mathcal{L}_k$ does not contain a non-trivial proper $SL(2,\mathbb{R})$-equivariant subbundle, i.e. that the restriction of the Kontsevich-Zorich cocycle to each $\mathcal{L}_k$ is strongly irreducible (see Definition C.2). If U has nontrivial intersection with the kernel of p, then we let $\lambda_0 = 0$, and let $\tilde{\Lambda} = \hat{\Lambda} \cup \{0\}$.

The Forni subbundle. Let

$$F(x) = \bigcup_{\{k \,:\, \hat{\lambda}_k = 0\}} \mathcal{L}_k(x).$$

We call $F(x)$ the *Forni* subspace of μ. The subspaces $F(x)$ form a subbundle of the Hodge bundle which we call the Forni subbundle. It is an $SL(2,\mathbb{R})$-invariant subbundle, on which the Kontsevich-Zorich cocycle acts by Hodge isometries. In particular, all the Lyapunov exponents of $F(x)$ are 0. Let $F^\perp(x)$ denote the orthogonal complement to $F(x)$ in the Hodge norm. By Theorem A.9 (b),

$$F^\perp(x) = \bigcup_{\{k \,:\, \hat{\lambda}_k \neq 0\}} \mathcal{L}_k(x).$$

The following is proved in [AEM]:

Theorem 14.3. *There exists a subset Φ of the stratum with $\nu(\Phi) = 1$ such that for all $x \in \Phi$ there exists a neighborhood $\mathcal{U}(x)$ such that for all $y \in \mathcal{U}(x) \cap \Phi$ we have $p(y - x) \in F^{\perp}(x)$.*

The backwards shift map. Let B be the space of (one-sided) infinite sequences of elements of $SL(2, \mathbb{R})$. (We think of B as giving the "past" trajectory of the random walk.) Let $T : B \to B$ be the shift map. (In our interpretation, T takes us one step into the past). We define the skew-product map $T : B \times X \to B \times X$ by

$$T(b, x) = (Tb, b_0^{-1}x), \qquad \text{where } b = (b_0, b_1, \dots)$$

(Thus the shift map and the skew-product map are denoted by the same letter.) We define the measure β on B to be $\mu \times \mu \dots$.

For each $k \in \hat{\Lambda}$, we have the Lyapunov flag for T

$$\{0\} = V_0^{(k)} \subset V_1^{(k)}(b, x) \subset \dots V_{n_k}^{(k)}(b, x) = \mathcal{L}_k(x).$$

The two-sided shift space. Let $\tilde{B}$ denote the two-sided shift space. We denote the measure $\dots \times \mu \times \mu \times \dots$ on $\tilde{B}$ also by β.

Notation. For $a, b \in B$ let

$$a \vee b = (\dots, a_2, a_1, b_0, b_1, \dots) \in \tilde{B}. \tag{14.3}$$

If $\omega = a \vee b \in \tilde{B}$, we think of the sequence

$$\dots, \omega_{-2}, \omega_{-1} = \dots a_2, a_1$$

as the "future" of the random walk trajectory. (In general, following [BQ], we use the symbols b, b' etc. to refer to the "past" and the symbols a, a' etc. to refer to the "future").

The opposite Lyapunov flag. Note that on the two-sided shift space $\tilde{B} \times X$, the map T is invertible. Thus, for each $a \vee b \in B$, we have the Lyapunov flag for T^{-1}:

$$\{0\} = \hat{V}_0^{(k)} \subset \hat{V}_1^{(k)}(a, x) \subset \dots \hat{V}_{n_k}^{(k)}(a, x) = \mathcal{L}_k(x),$$

(As reflected in the above notation, this flag depends only on the "future" i.e. "a" part of $a \vee b$).

The top Lyapunov exponent $\hat{\lambda}_k$. Let $\hat{\lambda}_k \geq 0$ denote the top Lyapunov exponent in $\mathcal{L}_k$. Then, (since T steps into the past), for $v \in V_1^{(k)}(b, x)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\|T^n(b, x)_* v\|}{\|v\|} = -\hat{\lambda}_k \tag{14.4}$$

In the above equation we used the notation $T^n(b, x)_*$ to denote the action of $T^n(b, x)$ on $H^1(M, \mathbb{R})$.

Also, for $v \in \hat{V}^{(k)}_{n_k-1}(a,x)$, for some $\alpha > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|T^{-n}(a \vee b, x)_* v\|}{\|v\|} < \hat{\lambda}_k - \alpha.$$

Here, α is the minimum over k of the difference between the top Lyapunov exponent in $\mathcal{L}_k$ and the next Lyapunov exponent.

The following lemma is a consequence of the zero-one law Lemma C.9(i):

Lemma 14.4. *For every $\delta > 0$ and every $\delta' > 0$ there exists $E_{good} \subset X$ with $\nu(E_{good}) > 1 - \delta$ and $\sigma = \sigma(\delta, \delta') > 0$, such that for any $x \in E_{good}$, any k and any vector $w \in \mathbb{P}^1(\mathcal{L}_k(x))$,*

$$\beta\left(\{a \in B \ : \ d(w, \hat{V}^{(k)}_{n_k-1}(a,x)) > \sigma\}\right) > 1 - \delta' \tag{14.5}$$

(In (14.5), $d(\cdot,\cdot)$ is some distance on the projective space $\mathbb{P}^1(H^1(M,\mathbb{R}))$).

Proof. It is enough to prove the lemma for a fixed k. For $F \subset Gr_{n_k-1}(\mathcal{L}_k(x))$ (the Grassmanian of $n_k - 1$ dimensional subspaces of $\mathcal{L}_k(x)$) let

$$\hat{\nu}^{(k)}_x(F) = \beta\left(\{a \in B \ : \ \hat{V}^{(k)}_{n_k-1}(a,x) \in F\}\right),$$

and let $\hat{\nu}^{(k)}$ denote the measure on the bundle $X \times Gr_{n_k-1}(\mathcal{L}_k)$ given by

$$d\hat{\nu}^{(k)}(x,L) = d\nu(x)\, d\hat{\nu}^{(k)}_x(L).$$

Then, $\hat{\nu}^{(k)}$ is a stationary measure for the (forward) random walk. For $w \in \mathbb{P}^1(\mathcal{L}_k(x))$ let $I(w) = \{L \in Gr_{n_k-1}(\mathcal{L}_k(x)) \ : \ w \in L\}$. Let

$$Z = \{x \in X \ : \ \hat{\nu}^{(k)}_x(I(w)) > 0 \text{ for some } w \in \mathbb{P}^1(\mathcal{L}_k(x))\},$$

Suppose $\nu(Z) > 0$. Then, for each $x \in Z$ we can choose $w_x \in \mathbb{P}^1(\mathcal{L}_k(x))$ such that $\hat{\nu}^{(k)}_x(I(w_x)) > 0$. Then,

$$\hat{\nu}^{(k)}\left(\bigcup_{x\in Z} \{x\} \times I(w_x)\right) > 0. \tag{14.6}$$

Therefore, (14.6) holds for some ergodic component of $\hat{\nu}^{(k)}$. However, this contradicts Lemma C.9 (i), since by the definition of $\mathcal{L}_k$, the action of the cocycle on $\mathcal{L}_k$ is strongly irreducible. Thus, $\nu(Z) = 0$ and $\nu(Z^c) = 1$. By definition, for all $x \in Z^c$ and all $w \in \mathcal{L}_k(x)$,

$$\beta\left(\{a \in B \ : \ w \in \hat{V}^{(k)}_{n_k-1}(a,x)\}\right) = 0.$$

Fix $x \in Z^c$. Then, for every $w \in \mathbb{P}^1(\mathcal{L}_k(x))$ there exists $\sigma_0(x,w,\delta') > 0$ such that

$$\beta\left(\{a \in B \ : \ d(\hat{V}^{(k)}_{n_k-1}(a,x), w) > 2\sigma_0(x,w,\delta')\}\right) > 1 - \delta'.$$

Let $\mathcal{U}(x,w) = \{z \in \mathbb{P}^1(\mathcal{L}_k(x)) \;:\; d(z,w) < \sigma_0(x,w,\delta')\}$. Then the $\{\mathcal{U}(x,w)\}_{w\in\mathbb{P}^1(\mathcal{L}_k(x))}$ form an open cover of the compact space $\mathbb{P}^1(\mathcal{L}_k(x))$, and therefore there exist $w_1, \dots w_n$ with $\mathbb{P}^1(\mathcal{L}_k(x)) = \bigcup_{i=1}^n \mathcal{U}(x,w_i)$. Let $\sigma_1(x,\delta') = \min_i \sigma_0(x,w_i,\delta')$. Then, for all $x \in Z^c$,

$$\beta\left(\{a \in B \;:\; d(\hat{V}^{(k)}_{n_k-1}(a,x), w) > \sigma_1(x,\delta')\}\right) > 1-\delta'.$$

Let $E_N(\delta') = \{x \in Z^c \;:\; \sigma_1(x,\delta') > \frac{1}{N}\}$. Since $\bigcup_{N=1}^\infty E_N(\delta') = Z^c$ and $\nu(Z^c) = 1$, there exists $N = N(\delta,\delta')$ such that $\nu(E_N(\delta')) > 1-\delta$. Let $\sigma = 1/N$ and let $E_{good} = E_N$. □

Lyapunov subspaces and Relative Homology. The following Lemma is well known:

Lemma 14.5. *The Lyapunov spectrum of the Kontsevich-Zorich cocycle acting on relative homology is the the Lyapunov spectrum of the Kontsevich-Zorich cocycle acting on absolute homology, union n zeroes, where $n = \ker p$.*

Let $\bar{\mathcal{L}}_k = p^{-1}(\mathcal{L}_k) \subset H^1(M,\Sigma,\mathbb{R})$. We have the Lyapunov flag

$$\{0\} = \bar{V}^{(k)}_0 \subset \bar{V}^{(k)}_1(b,x) \subset \dots \bar{V}^{(k)}_{n_k}(b,x) = \bar{\mathcal{L}}_k(x),$$

corresponding to the action on the invariant subspace $\bar{\mathcal{L}}_k \subset H^1(M,\Sigma,\mathbb{R})$. Also for each $a \in B$, we have the opposite Lyapunov flag

$$\{0\} = \hat{\bar{V}}^{(k)}_0 \subset \hat{\bar{V}}^{(k)}_1(a,x) \subset \dots \hat{\bar{V}}^{(k)}_{\bar{n}_k}(a,x) = \bar{\mathcal{L}}_k(x),$$

Lemma 14.6. *Suppose $\hat{\lambda}_k \neq 0$. Then for almost all (b,x),*

$$p(\bar{V}^{(k)}_1(b,x)) = V^{(k)}_1(b,x),$$

and p is an isomorphism between these two subspaces. Similarly, for almost all (a,x),

$$\hat{\bar{V}}^{(k)}_{\bar{n}_k-1}(a,x) = p^{-1}(V^{(k)}_{n_k-1}(a,x)).$$

Proof. In view of Lemma 14.5 and the assumption that $\hat{\lambda}_k \neq 0$, $\hat{\lambda}_k$ is the top Lyapunov exponent on both $\mathcal{L}_k$ and $\bar{\mathcal{L}}_k$. Note that

$$\bar{V}^{(k)}_1 = \{\bar{v} \in \bar{\mathcal{L}}_k \;:\; \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|T^n \bar{v}\|}{\|\bar{v}\|} \le -\hat{\lambda}_k.\} \tag{14.7}$$

Also,

$$V^{(k)}_1 = \{v \in \mathcal{L}_k \;:\; \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|T^n v\|}{\|v\|} \le -\hat{\lambda}_k.\} \tag{14.8}$$

It is clear from the definition of the Hodge norm on relative cohomology (A.1) that $\|p(v)\| \le C\|v\|$ for some absolute constant C. Therefore, it follows from (14.8) and (14.7) that $p(\bar{V}^{(k)}_1) \subset V^{(k)}_1$. But by Lemma 14.5, $\dim(\bar{V}^{(k)}_1) = \dim(V^{(k)}_1)$. Therefore, $\bar{V}^{(k)}_1 = V^{(k)}_1$. □

Remark. Even though we will not use this, a version Lemma 14.6 holds for all non-zero Lyapunov subspaces, and not just the subspace corresponding to the top Lyapunov exponent $\hat{\lambda}_k$.

The action on $H^1(M, \Sigma, \mathbb{C})$. By the multiplicative ergodic theorem applied to the action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$, for β-almost all $b \in B$,

$$\sigma_0 = \lim_{n \to \infty} \frac{1}{n} \log \|b_0 \dots b_n\|$$

where $\sigma_0 > 0$ is the Lyapunov exponent for the measure μ on $SL(2, \mathbb{R})$. Also, by the multiplicative ergodic theorem, for β-almost all $b \in B$ there exists a one-dimensional subspace $W_+(b) \subset \mathbb{R}^2$ such that $v \in W_+(b)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|b_n^{-1} \dots b_0^{-1} v\| = -\sigma_0.$$

For $x = (M, \omega)$ let $i_x : \mathbb{R}^2 \to H^1(M, \mathbb{R})$ denote the map $(a, b) \to a \operatorname{Re} x + b \operatorname{Im} x$. Let $W_+^{\perp}(b, x) \subset H^1(M, \Sigma, \mathbb{R})$ be defined by

$$W_+^{\perp}(b, x) = \{v \in H^1(M, \Sigma, \mathbb{R}) \ : \ p(v) \wedge w = 0 \text{ for all } w \in i_x(W_+(b)), \}$$

and let

$$W^+(b, x) = W_+(b) \otimes H^1(M, \Sigma, \mathbb{R}).$$

Since we identify $\mathbb{R}^2 \otimes H^1(M, \Sigma, \mathbb{R})$ with $H^1(M, \Sigma, \mathbb{C})$, we may consider $W^+(b, x)$ as a subspace of $H^1(M, \Sigma, \mathbb{C})$. This is the "stable" subspace for T. (Recall that T moves into the past).

For a "future trajectory" $a \in B$, we can similarly define a 1-dimensional subspace $W_-(a) \subset \mathbb{R}^2$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \|a_n \dots a_1 v\| = -\sigma_0 \qquad \text{for } v \in W_-(a).$$

Let $A : SL(2, \mathbb{R}) \times X \to Hom(H^1(M, \Sigma, \mathbb{R}), H^1(M, \Sigma, \mathbb{R}))$ denote the Kontsevich-Zorich cocycle. We then have the cocycle

$$\hat{A} : SL(2, \mathbb{R}) \times X \to Hom(H^1(M, \Sigma, \mathbb{C}), H^1(M, \Sigma, \mathbb{C}))$$

given by

$$\hat{A}(g, x)(v \otimes w) = gv \otimes A(g, x)w$$

and we have made the identification $H^1(M, \Sigma, \mathbb{C}) = \mathbb{R}^2 \otimes H^1(M, \Sigma, \mathbb{R})$. This cocycle can be thought of as the derivative cocycle for the action of $SL(2, \mathbb{R})$. From the definition we see that the Lyapunov exponents of $\hat{A}$ are of the form $\pm\sigma_0 + \lambda_i$, where the λ_i are the Lyapunov exponents of A.

15. Time changes and suspensions

There is a natural "forgetful" map $f : \tilde{B}^X \to B^X$. We extend functions on B^X to $\tilde{B}^X$ by making them constant along the fibers of f. The measure $\beta \times \nu$ is a T-invariant measure on $\tilde{B} \times X$.

The cocycles θ_j. By Theorem A.6, the restriction of the Kontsevich-Zorich cocycle to each $\mathcal{L}_j$ is semisimple (in fact simple). Then by Theorem C.5, the Lyapunov spectrum of T on each $\mathcal{L}_j$ is semisimple, and the restriction of T to the top Lyapunov subspace of each $\mathcal{L}_j$ consists of a single conformal block. This means that there is a inner product $\langle \, , \rangle_{j,b,x}$ defined on $W_+(b) \otimes V_1^{(j)}(b,x)$ and a cocycle $\theta_j : B \times X \to \mathbb{R}$ such that for all $u, v \in W_+(b) \otimes V_1^{(j)}(b,x)$,

$$\langle \hat{A}(b_0^{-1}, x)u, \hat{A}(b_0^{-1}, x)v\rangle_{j,Tb,b_0^{-1}x} = e^{-\theta_j(b,x)}\langle u, v\rangle_{j,b,x}, \tag{15.1}$$

To handle relative homology, we need to also consider the case in which the action of $A(\cdot,\cdot)$ is trivial. We thus define an inner product $\langle \, , \rangle_{0,b}$ on $\mathbb{R}^2$, and a cocycle $\theta_0 : B \to \mathbb{R}$ so that for $u, v \in W_+(b)$,

$$\langle b_0^{-1}u, b_0^{-1}v\rangle_{0,Tb} = e^{-\theta_0(b)}\langle u, v\rangle_{0,b}, \tag{15.2}$$

For notational simplicity, we let $\theta_0(b,x) = \theta_0(b)$.

Switch to positive cocycles. The cocycle θ_j corresponds to the $\hat{A}(\cdot,\cdot)$-Lyapunov exponent $\sigma_0 + \hat{\lambda}_j$, where $\hat{\lambda}_j$ is the top Lyapunov exponent of $A(\cdot,\cdot)$ in $\mathcal{L}_j$. Since $\hat{\lambda}_j \geq 0$,

$$\sigma_0 + \hat{\lambda}_j = \int_{B\times X} \theta_j(b,x)\, d\beta(b)\, d\nu(x) > 0.$$

Thus, the cocycle θ_j has positive average on $B \times X$. However, we do not know that θ_j is positive, i.e. that for all $(b,x) \in B \times X$, $\theta_j(b,x) > 0$. This makes it awkward to use $\theta_j(b,x)$ to define a time change. Following [BQ] we use a positive cocycle τ_j equivalent to θ_j.

By [BQ, Lemma 2.1], we can find a *positive* cocycle $\tau_j : B \times X \to \mathbb{R}$ and a measurable function $\phi_j : B \times X \to \mathbb{R}$ such that

$$\theta_j - \phi_j \circ T + \phi_j = \tau_j \tag{15.3}$$

and

$$\int_{B\times X} \tau_j(b,x)\, d\beta(b)\, d\nu(x) < \infty.$$

For $v \in W_+(b) \otimes V_1^{(j)}(b,x)$ we define

$$\|v\|'_{j,b,x} = e^{\phi_j(b,x)}\|v\|_{j,b,x}. \tag{15.4}$$

where the norm $\langle \cdot,\cdot\rangle_j$ is as in (15.1) and (15.2). Then

$$\|\hat{A}(b_0^{-1}, x)v\|'_{j,T(b,x)} = e^{-\tau_j(b,x)}\|v\|'_{j,b,x}. \tag{15.5}$$

Suspension. Let $B^X = B \times X \times (0,1]$. Recall that β denotes the measure on B which is given by $\mu \times \mu \dots$. Let β^X denote the measure on B^X given by $\beta \times \nu \times dt$, where dt is the Lebesgue measure on $(0,1]$. In B^X we identify $(b,x,0)$ with $(T(b,x),1)$, so that B^X is a suspension of T. We can then define a suspension flow $T_t : B^X \to B^X$ in the natural way. Then T_t preserves the measure β^X.

Let $T_t(b,x,s)_*$ denote the action of $T_t(b,x,s)$ on $H^1(M,\Sigma,\mathbb{C})$ (i.e. the derivative cocycle on the tangent space). Then, for $t \in \mathbb{Z}$ and $v \in W_+(b) \otimes V_1^{(j)}(b,x)$ and $0 < s \le 1$ we have, in view of (15.5),

$$\|T_t(b,x,s)_* v\|'_{j,T(b,x)} = e^{-\tau_j(t,b,x)} \|v\|'_{j,b,x}, \tag{15.6}$$

where $\tau_j(t,b,x) = \sum_{n=0}^{t-1} \tau_j(T^n(b,x))$. We can extend the norm $\|\cdot\|'_j$ from $B \times X$ to B^X by

$$\|v\|'_{j,b,x,s} = \|v\|'_{j,b,x} e^{-s\tau_j(b,x)}. \tag{15.7}$$

Then (15.6) holds for all $t \in \mathbb{R}$ provided we set for $n \in \mathbb{Z}$ and $0 \le s < 1$,

$$\tau_j(n+s,b,x) = \tau_j(n,b,x) + s\tau_j(T^n(b,x)).$$

The time change. Here we differ slightly from [BQ] since we would like to have several different time changes of the flow T_t on the same space. Hence, instead of changing the roof function, we keep the roof function constant, but change the speed in which one moves on the $[0,1]$ fibers.

Let $T_t^{\tau_j} : B^X \to B^X$ be the time change of T_t where on $(b,x) \times [0,1]$ one moves at the speed $\tau_j(b,x)$. More precisely, we set

$$T_t^{\tau_j}(b,x,s) = (b,x,s - \tau_j(b,x)t), \qquad \text{if } 0 < s - \tau_j(b,x)t \le 1, \tag{15.8}$$

and extend using the identification $((b,x),0) = (T(b,x),1)$.

Then $T_\ell^{\tau_k}$ is the operation of moving backwards in time far enough so that the cocycle multiplies the direction of the top Lyapunov exponent in $\mathcal{L}_k$ by $e^{-\ell}$. In fact, by (15.6) and (15.8), we have, for $v \in W_+(b) \otimes V_1^{(k)}(b,x)$,

$$\|T_\ell^{\tau_k}(b,x,s)_* v\|'_{j,T_\ell^{\tau_k}(b,x,s)} = e^{-\ell} \|v\|'_{j,b,x,s}. \tag{15.9}$$

The map T^{τ_k} and the two-sided shift space. On the space $\tilde{B}^X$, T^{τ_k} is invertible, and we denote the inverse of $T_\ell^{\tau_k}$ by $T_{-\ell}^{\tau_k}$. We write

$$T_{-\ell}^{\tau_k}(a \vee b, x, s)_* \tag{15.10}$$

for the linear map on the tangent space $H^1(M,\Sigma,\mathbb{C})$ induced by $T_{-\ell}^{\tau_k}(a \vee b, x, s)$. In view of (15.4) and (15.9), we have for $v \in W_+(b) \otimes V_1^{(k)}(b,x)$,

$$\|T_{-\ell}^{\tau_k}(a \vee b, x, s)_* v\| = \exp(\ell - \phi_k(b,x,s) + \phi_k(T_{-\ell}^{\tau_k}(a \vee b, x, s)))\|v\|. \tag{15.11}$$

Here we have omitted the subscripts on the norm $\|\cdot\|_{k,b,x}$ and also extended the function $\phi_k(b,x,s)$ so that for all $(b,x,s) \in B^X$ and all $v \in W_+(b) \otimes V_1^{(k)}(b,x)$,

$$\|v\|_{k,b,x} = e^{\phi_k(b,x,s)} \|v\|'_{k,b,x,s}.$$

Invariant measures for the time changed flows. Let $\beta^{\tau_j,X}$ denote the measure on B^X given by

$$d\beta^{\tau_j,X}(b,x,t) = \frac{c_j}{\tau_j(b,x)} d\beta(b)\, d\nu(x)\, dt,$$

where the $c_j \in \mathbb{R}$ is chosen so that $\beta^{\tau_j,X}(B^X) = 1$. Then the measures $\beta^{\tau_j,X}$ are invariant under the flows $T_t^{\tau_j}$. We note the following trivial:

Lemma 15.1. *The measures $\beta^{\tau_j,X}$ are all absolutely continuous with respect to β^X. For every $\delta > 0$ there exists a compact subset $\mathcal{K} = \mathcal{K}(\delta) \subset B^X$ and $L = L(\delta) < \infty$ such that for all j,*

$$\beta^{\tau_j,X}(\mathcal{K}) > 1 - \delta,$$

and also for $(b,x,t) \in \mathcal{K}$,

$$\frac{d\beta^{\tau_j,X}}{d\beta^X}(b,x,t) \le L, \quad \frac{d\beta^X}{d\beta^{\tau_j,X}}(b,x,t) \le L.$$

Proof. Let

$$K_{j,n} = \{(b,x,t) \;\;:\;\; t \in [0,1], \quad \frac{1}{n} < \tau_j(b,x) < n\}.$$

Then, for all j, k,

$$\beta^{\tau_j,X}\left(\bigcup_{n=1}^{\infty} K_{k,n}\right) = 1.$$

It follows that there exists $N \in \mathbb{N}$ such that for all j,

$$\beta^{\tau_j,X}(K_{k,N}) \ge 1 - \delta/d,$$

where d is the maximal number of Lyapunov exponents j. Let

$$\mathcal{K} = \bigcap_k K_{k,N}.$$

Then, for all j, $\beta_{\tau_j,X}(\mathcal{K}) \ge 1 - \delta$. We can now choose L so that $L > N^2 \max_j(c_j^2, c_j^{-2})$. $\square$

16. The Martingale Convergence Argument

Standing Assumptions. We assume that the conditional measures of ν_b along $W^{\pm}(b,x)$ is supported on $U^{\pm}(b,x)$, and also that the conditional measures of ν_b along $U^{\pm}(b,x)$ are Lebesgue.

Lemma 16.1. *There exists a subset $\Psi \subset B^X$ with $\beta^X(\Psi) = 1$ such that for all $(b,x) \in \Psi$,*

$$\Psi \cap W^+(b,x) \cap \text{ ball of radius } 1 \ \subset \Psi \cap U^+(b,x).$$

Proof. See [MaT] or [EL, 6.23]. $\square$

The parameter δ. Let $\delta > 0$ be a parameter which will eventually be chosen sufficiently small. We use the notation $c_i(\delta)$ and $c_i'(\delta)$ for functions which tend to 0 as $\delta \to 0$. In this section we use the notation $A \approx B$ to mean that the ratio A/B is bounded between two positive constants depending on δ.

We first choose a compact subset $K_0 \subset \Psi \cap \Phi$ with $\beta^X(K_0) > 1 - \delta > 0.999$, the conull set Ψ is as in Lemma 16.1, and the conull set Φ is as in Theorem 14.3. By the multiplicative ergodic theorem and (14.4), we may also assume that there exists $\ell_1(\delta) > 0$ such that for all $(b,x,s) \in K_0$ all k and all $v \in V_1^{(k)}(b,x)$ and all $\ell > \ell_1(\delta)$,

$$\|T_\ell(b,x,s)_* v\| \le e^{-(\lambda_k/2)\ell}\|v\|. \tag{16.1}$$

(Here, as in (14.4) the notation $T_\ell(b,x,s)_*$ denotes the action on $H^1(M,\Sigma,\mathbb{R})$.)

Lemma 16.2. *For every $\delta > 0$ there exists $K \subset B^X$ and $C = C(\delta) < \infty$, $\beta = \beta(\delta) > 0$ and $C' = C'(\delta) < \infty$ such that*

(K1) *For all $L > C'(\delta)$, and all $(b,x,s) \in K$,*

$$\frac{1}{L}\int_0^L \chi_{K_0}(T_t(b,x,s))\,dt \ge 0.99$$

(K2) *$\beta^X(K) > 1 - c_1(\delta)$. Also, for all j, $\beta^{\tau_j,X}(K) > 1 - c_1(\delta)$.*

(K3) *For all j and all $(b,x,t) \in K$, $|\phi_j(b,x,t)| < C$, where ϕ_j is as in (15.3).*

(K4) *For all j, all $(b,x,t) \in K$ all $k \neq 0$ and all $v \in \bar{V}_1^{(k)}(b,x)$,*

$$\|p(v)\| \ge \beta(\delta)\|v\|. \tag{16.2}$$

Proof. By the Birkhoff ergodic theorem, there exists $\mathcal{K}'' \subset B^X$ such that $\beta^X(\mathcal{K}'') > 1-\delta/5$ and (K1) holds for $\mathcal{K}''$ instead of $\mathcal{K}$. We can choose $\mathcal{K}' \subset B^X$ and $C = C(\delta) < \infty$ such that $\beta^X(\mathcal{K}') > 1-\delta/5$ and (K3) holds for $\mathcal{K}'$ instead of K. Let $\mathcal{K} = \mathcal{K}(\delta/5)$ and $L = L(\delta/5)$ be as in Lemma 15.1 with $\delta/5$ instead of δ. Then choose $K_j \subset \Psi$ with $\beta^{\tau_j,X}(K_j) > 1-\delta/(5dL)$, where d is the number of Lyapunov exponents. In view of Lemma 14.6 there exists $\mathcal{K}''' \subset X$ with $\beta^X(\mathcal{K}''') > 1-\delta/5$ so that (16.2) holds. Then, let $K = \mathcal{K}''' \cap \mathcal{K}'' \cap \mathcal{K}' \cap \mathcal{K} \cap \bigcap_j K_j$. The properties (K1), (K2), (K3) and (K4) are easily verified. $\square$

Warning. In the rest of this section, we will often identify K and K_0 with their pullbacks $f^{-1}(K) \subset \tilde{B}^X$ and $f^{-1}(K_0) \subset \tilde{B}^X$ where $f : \tilde{B}^X \to B^X$ is the forgetful map.

The Martingale Convergence Theorem. Let $\mathcal{B}^{\tau_j,X}$ denote the σ-algebra of $\beta^{\tau_j,X}$ measurable functions on B^X. As in [BQ], let

$$Q_\ell^{\tau_j,X} = (T_\ell^{\tau_j})^{-1}(\mathcal{B}^{\tau_j,X}),$$

(Thus if a function F is measurable with respect to $Q_\ell^{\tau_j,X}$, then F depends only on what happened at least ℓ time units in the past, where ℓ is measured using the time change τ_j.)

Let

$$Q_\infty^{\tau_j,X} = \bigcap_{\ell>0} Q_\ell^{\tau_j,X}.$$

The $Q_\ell^{\tau_j,X}$ are a decreasing family of σ-algebras, and then, by the Martingale Convergence Theorem, for $\beta^{\tau_j,X}$-almost all $(b,x,s) \in B^X$,

$$\lim_{\ell\to\infty} \mathbb{E}_j(1_K \mid Q_\ell^{\tau_j,X})(b,x,s) = \mathbb{E}_j(1_K \mid Q_\infty^{\tau_j,X})(b,x,s) \tag{16.3}$$

where $\mathbb{E}_j$ denotes expectation with respect to the measure $\beta^{\tau_j,X}$.

The set S'. In view of (16.3) and the condition (K2) we can choose $S' = S'(\delta) \subset B^X$ be such that for all $\ell > \ell_0$, all j, and all $(b,x,s) \in S'$,

$$\mathbb{E}_j(1_K \mid Q_\ell^{\tau_j,X})(b,x,s) > 1 - c_2(\delta). \tag{16.4}$$

By using Lemma 15.1 as in the proof of Lemma 16.2 we may assume that (by possibly making ℓ_0 larger) we have for all j,

$$\beta^{\tau_j,X}(S') > 1 - c_2(\delta). \tag{16.5}$$

The set E_{good}. By Lemma 14.4 we may choose a subset $E_{good} \subset \tilde{B}^X$ (which is actually of the form $\tilde{B} \times E'_{good}$ for some subset $E'_{good} \subset X \times [0,1]$), with $\beta^X(E_{good}) > 1 - c_3(\delta)$, and a number $\sigma(\delta) > 0$ such that for any $(b,x,s) \in E_{good}$, any j and any unit vector $w \in \mathcal{L}_j(b,x)$,

$$\beta\left(\{a \in B \;\;:\;\; d(w, \hat{V}^{(j)}_{n_j-1}(a,x)) > \sigma(\delta)\}\right) > 1 - c_3'(\delta). \tag{16.6}$$

We may assume that $E_{good} \subset K$. By the Osceledec multiplicative ergodic theorem and Lemma 14.6, we may also assume that there exists $\alpha > 0$ (depending only on the Lyapunov spectrum), and $\ell_0 = \ell_0(\delta)$ such that for $(b,x,s) \in E_{good}$, $\ell > \ell_0$, at least $1 - c_3''(\delta)$ measure of $a \in B$, and all $\bar{v} \in \hat{\hat{V}}^{(j)}_{n_j-1}(a,x)$,

$$\|T^{\tau_j}_{-\ell}(a \vee b, x, s)_* v\| \le e^{(1-\alpha)\ell}\|\bar{v}\|. \tag{16.7}$$

The sets Ω_ρ. In view of (16.5) and the Birkhoff ergodic theorem, for every $\rho > 0$ there exists a set $\Omega_\rho = \Omega_\rho(\delta) \subset \tilde{B}^X$ such that

(Ω1) $\beta^X(\Omega_\rho) > 1 - \rho$.

(Ω2) There exists $\ell_0' = \ell_0'(\rho)$ such that for all $\ell > \ell_0'$, and all $(b, x, s) \in \Omega_\rho$,

$$|\{t \in [-\ell, \ell] \ : \ T_t(b, x, s) \in S' \cap E_{good}\}| \geq (1 - c_5(\delta))2\ell.$$

Lemma 16.3. *Suppose the measure ν is not affine. Then there exists $\rho > 0$ so that for every $\delta' > 0$ there exist $(b, x, s) \in \Omega_\rho$, $(b, y, s) \in \Omega_\rho$ with $\|y - x\| < \delta'$ such that $p(y - x) \in p(U^\perp)_{\mathbb{C}}(x)$,*

$$d(y - x, U_{\mathbb{C}}(x)) > \frac{1}{10}\|y - x\| \tag{16.8}$$

and

$$d(y - x, W^+(b, x)) > \frac{1}{3}\|y - x\| \tag{16.9}$$

(so $y - x$ is in general position with respect to $W^+(b, x)$.)

Proof. Recall that we are assuming that ν is $SL(2, \mathbb{R})$ invariant. Therefore, $\nu_b = \nu$ for almost all $b \in B$. Hence, the measure β^X is the product measure $\beta \times \nu \times dt$. Then, by Fubini's theorem, there exists a subset $\Omega_\rho' \subset X$ with $\nu(\Omega_\rho') \geq 1 - \rho^{1/2}$ such that for $x \in \Omega_\rho'$,

$$(\beta \times dt)(\{(b, s) \ : \ (b, x, s) \in \Omega_\rho\}) \geq (1 - \rho^{1/2}). \tag{16.10}$$

We decompose X into disjoint local charts, so up to a null set,

$$X = \bigsqcup_\alpha J_\alpha,$$

and for all α there exists some some $\delta_\alpha > 0$, such that for all $x \in J_\alpha$,

$$\mathrm{Vol}(U_{\mathbb{C}}[x] \cap J_\alpha) \geq \delta_\alpha, \tag{16.11}$$

where $\mathrm{Vol}(\cdot)$ is as in Lemma 14.1. Let

$$\Omega_\rho'' = \{x \in J_\alpha \ : \ \nu_{U_{\mathbb{C}}(x)}(\Omega_\rho' \cap J_\alpha) \geq (1 - \rho^{1/4})\nu_{U_{\mathbb{C}}(x)}(J_\alpha)\}. \tag{16.12}$$

In the above equation, $\nu_{U_{\mathbb{C}}(x)}$ is the conditional measure of ν along $U_{\mathbb{C}}[x]$ (which is in fact a multiple of Lebesgue measure). By Fubini, $\nu(\Omega_\rho'') \geq (1 - \rho^{1/4})$. In particular, $\bigcup_{\rho > 0} \Omega_\rho''$ is conull.

Note that by the definition of Ω_ρ'', if $x \in \Omega_\rho'' \cap J_\alpha$ then $U_{\mathbb{C}}[x] \cap J_\alpha \subset \Omega_\rho''$. It follows that we may write, for some indexing set $I_\alpha(\rho)$,

$$\Omega_\rho'' \cap J_\alpha = \bigsqcup_{x \in I_\alpha(\rho)} U_{\mathbb{C}}[x] \cap J_\alpha.$$

Suppose that for all α and all $\rho > 0$, $I_\alpha(\rho)$ is countable. Then, for a positive measure set of x,

$$\nu(U_{\mathbb{C}}[x] \cap J_\alpha) > 0. \tag{16.13}$$

Then by ergodicity, (16.13) holds for ν-almost all $x \in X$ and without loss of generality, for all $x \in I_\rho(\alpha)$. The restriction of ν to $U_{\mathbb{C}}[x]$ is a multiple of Lebesgue measure, therefore there exists a constant $\psi(x) \neq 0$ such that for $E \subset U_{\mathbb{C}}[x]$, $\nu(E) = \psi(x)\,\mathrm{Vol}(E)$, where Vol is as defined in Lemma 14.1. Since both ν and Vol are invariant under the $SL(2,\mathbb{R})$ action, $\psi(x)$ is invariant, and thus by ergodicity ψ is constant almost everywhere.

Let $I'_\alpha = \bigcup_{\rho>0} I_\alpha(\rho)$. For $x, y \in I'_\alpha$ write $x \sim y$ if $U_{\mathbb{C}}[x] \cap J_\alpha = U_{\mathbb{C}}[y] \cap J_\alpha$, and let $I''_\alpha \subset I'_\alpha$ be the subset where we keep only one member of each $\sim$-equivalence class. Then (16.11) implies that for each α,

$$\nu(J_\alpha) \geq \sum_{x \in I''_\alpha} \nu(U_{\mathbb{C}}[x] \cap J_\alpha) = \sum_{x \in I''_\alpha} \psi\,\mathrm{Vol}(U_{\mathbb{C}}[x] \cap J_\alpha) \geq \psi \delta_\alpha |I''_\alpha|.$$

where $|\cdot|$ denotes the cardinality of a set. Since ν is a finite measure, we get that I''_α is finite, which implies that ν is affine.

Thus, we may assume that there exist α and $\rho > 0$ such that $I_\alpha(\rho)$ is not countable. Then we can find $x_1 \in I_\alpha(\rho)$ and $y_n \in I_\alpha(\rho)$ such that

$$\lim_{n\to\infty} hd(U_{\mathbb{C}}[x_1] \cap J_\alpha, U_{\mathbb{C}}[y_n] \cap J_\alpha) = 0,$$

where hd denotes Hausdorff distance between sets. Then, in view of the definition (16.12) of Ω''_ρ, for sufficiently large n, there exist $x \in U_{\mathbb{C}}[x_1] \cap \Omega'_\rho$ and $y \in U_{\mathbb{C}}[y_n] \cap \Omega'_\rho$ such that $y - x \in U_{\mathbb{C}}^\perp(x)$, and $\|y - x\| < \delta'$. Then, by the definition (16.10) of Ω'_ρ, we can choose (b, s) so that $(b, x, s) \in \Omega_\rho$, $(b, y, s) \in \Omega_\rho$, and (16.8) and (16.9) holds. $\square$

Standing Assumption. We fix $\rho = \rho(\delta)$ so that Lemma 16.3 holds.

The main part of the proof is the following:

Proposition 16.4. *There exists $C(\delta) > 1$ such that the following holds: Suppose for every $\delta' > 0$ there exist $(b, x, s), (b, y, s) \in \Omega_\rho$ with $\|x - y\| \leq \delta'$, $p(x - y) \in p(U_{\mathbb{C}}^\perp)(x)$, and so that (16.8) and (16.9) hold. Then for every $\epsilon > 0$ there exist $(b'', x'', s'') \in K$, $(b'', y'', s'') \in K$, such that $y'' - x'' \in U_{\mathbb{C}}^\perp(x'')$,*

$$\frac{\epsilon}{C(\delta)} \leq \|y'' - x''\| \leq C(\delta)\epsilon,$$

$$d(y'' - x'', U_{\mathbb{C}}(x'')) \geq \frac{1}{C(\delta)} \|y'' - x''\|, \tag{16.14}$$

$$d(y'' - x'', W^+(b'', x'')) < \delta'', \tag{16.15}$$

where δ'' depends only on δ', and $\delta'' \to 0$ as $\delta' \to 0$.

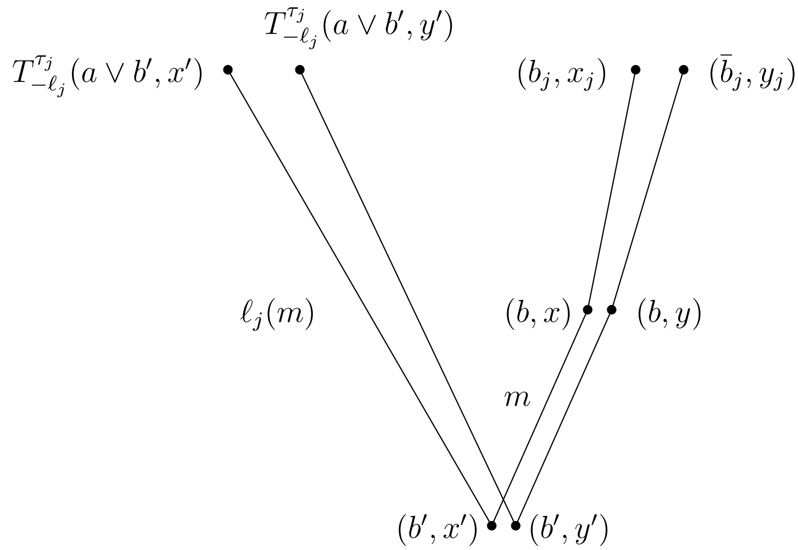

Figure 6. Proof of Proposition 16.4.

Proof. We can decompose

$$p(U^\perp(x)) = \bigoplus \mathcal{L}_k(x)$$

as in §14. Let π_j denote the projection to $\mathcal{L}_j$, using the decomposition (14.2). For $m \in \mathbb{R}^+$, write

$$(b', x', s') = T_m(b, x, s), \quad (b', y', s') = T_m(b, y, s).$$

and let

$$w_j(m) = \pi_j(x' - y')$$

(We will always have m small enough so that the above equation makes sense). Let $\ell_j(m)$ be such that

$$e^{\ell_j(m)} \|w_j(m)\| = \epsilon,$$

We also need to handle the relative homology part (where the action of the Kontsevich-Zorich cocycle is trivial). Set $\ell_0(m)$ to be the number such that

$$e^{\ell_0(m)} \|x' - y'\| = \epsilon.$$

Choose $0 < \sigma' \ll \lambda_{min}$ where $0 < \lambda_{min} = \min_{j \in \tilde{\Lambda}} \hat{\lambda}_j$. We will be choosing m so that

$$\frac{\sigma'}{2} |\log \|y - x\|| \le m \le \sigma' |\log \|y - x\||. \tag{16.16}$$

In view of (16.9) and Theorem A.1, (after some uniformly bounded time), $\|w_j(m)\|$ is an increasing function of m. Therefore, $\ell_j(m)$ is a decreasing function of m.

For a bi-infinite sequence $b \in \tilde{B}$ and $x \in X$, let

$$G_j(b, x, s) = \{m \in \mathbb{R}_+ \ : \ T^{\tau_j}_{-\ell_j(m)} T_m(b, x, s) \in S'\}$$

Let $G_{all}(b, x, s) = \bigcap_j G_j(b, x, s) \cap \{m \ : \ T_m(b, x, s) \in E_{good}\}$.

Lemma 16.5. *For $(b, x, s) \in \Omega_\rho$ and N sufficiently large,*

$$\frac{|G_{all}(b, x, s) \cap [0, N]|}{N} \geq 1 - c_6(\delta).$$

Proof. We can write $T^{\tau_j}_{-\ell_j(m)} T_m = T_{-g_j(m)}$. By definition,

$$m \in G_j(b, x, s) \quad \text{if and only if} \quad T_{-g_j(m)}(b, x, s) \in S'$$

Since $\ell_j(m)$ is a decreasing function of m, so is g_j, and in fact, for all $m_2 > m_1$

$$g_j(m_1) - g_j(m_2) > m_2 - m_1.$$

This implies that

$$g_j^{-1}(m_1) - g_j^{-1}(m_2) < m_1 - m_2. \tag{16.17}$$

Let $F = \{t \in [0, \dots, g_j(N)] \ : \ T_{-t}(b, x) \notin S'\}$. By condition ($\Omega$2), for N large enough, $|F| \leq (1 - c_5(\delta)) g_j(N)$. Note that $G_j^c \cap [0, N] = g_j^{-1}(F)$. Then, by (16.17),

$$|G_j^c \cap [0, N]| = |g_j^{-1}(F)| \leq |F| \leq c_5(\delta) g_j(N) \leq c_6(\delta) N,$$

where as in our convention $c_6(\delta) \to 0$ as $\delta \to 0$. □

We now continue the proof of Proposition 16.4. We may assume that δ' is small enough so that the right-hand-side of (16.16) is smaller then the N of Lemma 16.5. Suppose $(b, x, s) \in \Omega_\rho$, $(b, y, s) \in \Omega_\rho$. By Lemma 16.5, we can fix $m \in G_{all}(x)$ such that (16.16) holds. Write $\ell_j = \ell_j(m)$. Let

$$(b', x', s') = T_m(b, x, s), \quad (b', y', s') = T_m(b, y, s).$$

For $j \in \tilde{\Lambda}$, let

$$(b_j, x_j, s_j) = T^{\tau_j}_{-\ell_j(m)}(b', x', s'), \quad (\bar{b}_j, y_j, \bar{s}_j) = T^{\tau_j}_{-\ell_j(m)}(b', y', s').$$

Since $m \in G_{all}(b, x, s)$, we have $(b_j, x_j, s_j) \in S'$, $(\bar{b}_j, y_j, \bar{s}_j) \in S'$. Then, by (16.4), for all j,

$$\mathbb{E}_j(1_K \mid Q^{\tau_j, X}_{\ell_j})(b_j, x_j, s_j) > (1 - c_2(\delta)), \quad \mathbb{E}_j(1_K \mid Q^{\tau_j, X}_{\ell_j})(\bar{b}_j, y_j, \bar{s}_j) > (1 - c_2(\delta)).$$

Since $T^{\tau_j}_{\ell_j}(b_j, x_j, s_j) = (b', x', s')$, by [BQ, (7.5)] we have

$$\mathbb{E}_j(1_K \mid Q^{\tau_j, X}_{\ell_j})(b_j, x_j, s_j) = \int_B 1_K(T^{\tau_j}_{-\ell_j}(a \vee b', x', s'))\, d\beta(a),$$

where the notation $a \vee b'$ is as in (14.3). Thus, for all $j \in \tilde{\Lambda}$,

$$\beta\Big(\{a \ : \ T^{\tau_j}_{-\ell_j}(a \vee b', x', s') \in K\}\Big) > 1 - c_2(\delta). \tag{16.18}$$

Similarly, for all $j \in \tilde{\Lambda}$,

$$\beta\left(\{a \ : \ T^{\tau_j}_{-\ell_j}(a \vee b', y', s') \in K\}\right) > 1 - c_2(\delta).$$

Let $w = x' - y'$, and let $w_j = \pi_j(w)$. We can write

$$w = \bar{w}_0 + \sum_{j \in \hat{\Lambda}} \bar{w}_j \tag{16.19}$$

where $\bar{w}_0 \in \ker p$, and for $j > 0$, $\bar{w}_j$ are chosen so that $\pi_j(\bar{w}_j) = w_j$, and also $\|\bar{w}_j\| \approx \|w_j\|$.

For any $a \in B$, we may write

$$w_j = \xi_j(a) + v_j(a),$$

where $\xi_j(a) \in W_+(b') \otimes V_1^{(j)}(b', x')$, and

$$v_j(a) \in W_+(b) \otimes \hat{V}^{(j)}_{n_j - 1}(a, x') + W_-(a) \otimes \mathcal{L}_j(b', x').$$

This decomposition is motivated as follows: if we consider the Lyapunov decomposition

$$\mathbb{C} \otimes \mathcal{L}_j(x) = \bigoplus_k \mathcal{V}_k(a \vee b, x)$$

then $\xi_j(a)$ belongs to the subspace $\mathcal{V}_1(a \vee b, x)$ corresponding to the top Lyapunov exponent $\sigma_0 + \hat{\lambda}_j$ for the action of T_{-t}, and $v_j \in \oplus_{k \geq 2} \mathcal{V}_k(a \vee b, x)$ will grow with a smaller Lyapunov exponent under T_{-t}. Then $v_j(a)$ will also grow with a smaller Lyapunov exponent then $\xi_j(a)$ under $T^{\tau_j}_{-\ell}$.

Since $m \in G_{all}(b, x, s)$, we have $(b', x', s') \in E_{good}$. Then, by (16.6), for at least $1 - c_3'(\delta)$ fraction of $a \in B$,

$$\|v_j(a)\| \approx \|\xi_j(a)\| \approx \|w_j\| \approx \epsilon e^{-\ell_j}, \tag{16.20}$$

where the notation $A \approx B$ means that A/B is bounded between two constants depending only on δ. Since $(b', x', s') \in E_{good} \subset K$, by condition (K3) we have $|\phi_j(b', x', s')| \leq C(\delta)$. Also by (16.18), for at least $1 - c_2(\delta)$ fraction of $a \in B$, we have $T^{\tau_j}_{-\ell_j}(a \vee b', x', s') \in K$, so again by condition (K3) we have

$$|\phi_j(T^{\tau_j}_{-\ell_j}(a \vee b', x', s'))| \leq C(\delta).$$

Thus, by (16.20), (15.11) and (16.7), we have, for all $j \in \tilde{\Lambda}$, and at least $1 - c_4(\delta)$ fraction of $a \in B$,

$$\|T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* \xi_j(a)\| \approx \epsilon, \text{ and } \|T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* v_j(a)\| = O(e^{-\alpha \ell_j}), \tag{16.21}$$

where $\alpha > 0$ depends only on the Lyapunov spectrum. (The notation in (16.21) is defined in (15.10)). Hence, for at least $1 - c_4(\delta)$ fraction of $a \in B$,

$$\|T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* w_j\| \approx \epsilon,$$

Since $\lambda_j \geq 0$ (and by Theorem 14.3, if $\lambda_j = 0$ then $j = 0$, and $\bar{w}_0 \in \ker p$ where the action of the Kontsevich-Zorich cocycle is trivial), we have for at least $1 - c_4(\delta)$ fraction of $a \in B$,

$$\|T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* \bar{w}_j\| \approx \epsilon, \tag{16.22}$$

Let

$$t_j(a) = \sup\{t > 0 \ : \ \|T_{-t}(a \vee b', x', s')_* \bar{w}_j\| \leq \epsilon\},$$

and let $j(a)$ denote a $j \in \tilde{\Lambda}$ such that $t_j(a)$ is as small as possible as j varies over $\tilde{\Lambda}$. Then, if $j = j(a)$, then by (16.22),

$$\|T_{-t_j(a)}(a \vee b', x', s')_* \bar{w}_j\| \approx \|T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* \bar{w}_j\| \approx \epsilon. \tag{16.23}$$

Also, for at least $1 - c_4(\delta)$-fraction of $a \in B$, if $j = j(a)$ and $k \neq j$, then by (16.22),

$$\|T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* \bar{w}_k\| \leq C_1(\delta)\epsilon, \tag{16.24}$$

where $C_1(\delta)$ depends only on δ. Therefore, by (16.19), (16.23), and (16.24), for at least $1 - c_4(\delta)$-fraction of $a \in B$, if $j = j(a)$,

$$\|T^{\tau_j}_{-\ell_j}(a \vee b', x', s')_* (y' - x')\| \approx \epsilon. \tag{16.25}$$

We now choose $\delta > 0$ so that $c_4(\delta) + 2c_2(\delta) < 1/2$, and using (16.18) we choose $a \in B$ so that (16.25) holds, and also

$$T^{\tau_j}_{-\ell_j}(a \vee b', x', s') \in K, \qquad T^{\tau_j}_{-\ell_j}(a \vee b', y', s') \in K.$$

We may write

$$T^{\tau_j}_{-\ell_j}(a \vee b', x', s') = T_{-t}(a \vee b, x', s'), \qquad T^{\tau_j}_{-\ell_j}(a \vee b', y', s') = T_{-t'}(a \vee b, y', s')$$

Then, $|t' - t| = O(\delta)$. Therefore by condition (K1), there exists t'' with $|t'' - t| = O(\delta)$ such that

$$(b'', x'', s'') = T_{-t''}(a \vee b', x', s') \in K_0, \qquad (b'', y'', s'') = T_{-t''}(a \vee b', y', s') \in K_0.$$

Since $\|w\| \approx \epsilon e^{-\ell_j}$, and $\ell_j \to \infty$ as $\delta' \to 0$, we have $\|w\| = \|x' - y'\| \to 0$ as $\delta' \to 0$. Since $T_{-t''}$ does not expand the W^- components, the W^- component of $x'' - y''$ is bounded by the W^- component of $x' - y'$. Thus, the size of the W^- component of $x'' - y''$ tends to 0 as $\delta' \to 0$. Thus (16.15) holds.

It remains to prove (16.14). If

$$\|p(y'' - x'')\| \geq \frac{1}{C(\delta)}\|y'' - x''\| \tag{16.26}$$

then (16.14) holds since $p(y'' - x'') \in p(U(x'')^\perp)$. This automatically holds for the case where $|\Sigma| = 1$ (and thus, in particular, there are no marked points). If not, we may write

$$y'' - x'' = w''_+ + \bar{w}''_0$$

where $\|w''_+\| \le c(\delta)\|\bar{w}''_0\|$ and $\bar{w}''_0 \in \ker p$. We will need to rule out the case where $\bar{w}''_0$ is very close to $U^+(x'') \cap \ker p$. We will show that this contradicts the assumption (16.8).

Let w'_+, $\bar{w}'_0$ be such that

$$w''_+ = T_{-t''}(a \vee b, x', s')_* w'_+, \quad \bar{w}''_0 = T_{-t''}(a \vee b, x', s')_* \bar{w}'_0.$$

Then $y' - x' = w'_+ + \bar{w}'_0$ and in view of (16.1) and (16.20),

$$\|w'_+\| \le e^{-\lambda_{min} t''/2} \|\bar{w}'_0\| \approx e^{-\lambda_{min} t''/2} \|y' - x'\|.$$

Applying $T_{-m}(b, x', s')$ to both sides we get

$$y - x = w_+ + \bar{w}_0,$$

where $\bar{w}_0 \in \ker p$, and

$$\|w_+\| \le e^{2m} \|w'_+\| \le e^{2m - \frac{\lambda_{min} t''}{2}} \|x - y\|.$$

By (16.16), $2m - \frac{\lambda_{min} t''}{2} \le -\frac{\lambda_{min} t''}{4}$. Thus, $\|w_+\| \le (1/100)\|y - x\|$. Therefore, by (16.8), we have

$$d(\bar{w}_0, \ker p \cap U_{\mathbb{C}}(x)) > \frac{1}{20}\|w_0\|.$$

Since the action of the cocycle on $\ker p$ is trivial (and we have shown that in our situation the component in $\ker p$ dominates throughout the process), this implies

$$d(\bar{w}''_0, \ker p \cap U_{\mathbb{C}}(x'')) > \frac{1}{20}\|w''_0\| \ge \frac{1}{40}\|y'' - x''\|.$$

This, together with with the assumption that (16.26) does not hold, implies (16.14). $\square$

Proof of Theorem 1.4. It was already proved in Theorem 2.1 that ν is $SL(2, \mathbb{R})$-invariant. Now suppose ν is not affine. We can apply Lemma 16.3, and then iterate Proposition 16.4 with $\delta' \to 0$ and fixed ϵ and δ. Taking a limit along a subsequence we get points $(b_\infty, x_\infty, s_\infty) \in K_0$ and $(b_\infty, y_\infty, s_\infty) \in K_0$ such that $\|x_\infty - y_\infty\| \approx \epsilon$, $y_\infty \in W^+(b_\infty, x_\infty)$ and $y_\infty \in (U^\perp)^+(b_\infty, x_\infty)$. This contradicts Lemma 16.1 since $K_0 \subset \Psi$. Hence ν is affine. $\square$

A. Forni's results on the $SL(2, \mathbb{R})$ action

In this appendix, we summarize the results we use from the fundamental work of Forni [Fo]. The recent preprint [FoMZ] contains an excellent presentation of these ideas and also some additional results which we will use as well.

A.1. **The Hodge norm and the geodesic flow.** Fix a point S in $\mathcal{H}(\alpha)$; then S is a pair (M,ω) where ω is a holomorphic 1-form on M. Let $\|\cdot\|_{H,t}$ denote the Hodge norm on the surface $M_t = \pi(g_t S)$. Here, as above $\pi : \mathcal{H}(\alpha) \to \mathcal{T}_g$ is the natural map taking (M,ω) to M.

The following fundamental result is due to Forni [Fo, §2]:

Theorem A.1. *There exists a constant C depending only on the genus, such that for any $\lambda \in H^1(M,\mathbb{R})$ and any $t \geq 0$,*

$$\|\lambda\|_{H,t} \leq Ce^t \|\lambda\|_{H,0}.$$

If in addition λ is orthogonal to ω, and for some compact subset $\mathcal{K}$ of $\mathcal{M}_g$, the geodesic segment $[S, g_t S]$ spends at least half the time in $\pi^{-1}(\mathcal{K})$, then we have

$$\|\lambda\|_{H,t} \leq Ce^{(1-\alpha)t} \|\lambda\|_{H,0},$$

where $\alpha > 0$ depends only on $\mathcal{K}$.

The Hodge norm on relative cohomology. Let Σ denote the set of zeroes of ω. Let $p : H^1(M,\Sigma,\mathbb{R}) \to H^1(M,\mathbb{R})$ denote the natural map. We define a norm $\|\cdot\|'$ on the relative cohomology group $H^1(M,\Sigma,\mathbb{R})$ as follows:

$$\|\lambda\|' = \|p(\lambda)\|_H + \sum_{(z,w)\in\Sigma\times\Sigma} \left| \int_{\gamma_{z,w}} (\lambda - h) \right|, \tag{A.1}$$

where $\|\cdot\|_H$ denotes the Hodge norm on $H^1(M,\mathbb{R})$, h is the harmonic representative of the cohomology class $p(\lambda)$ and $\gamma_{z,w}$ is any path connecting the zeroes z and w. Since $p(\omega)$ and h represent the same class in $H^1(M,\mathbb{R})$, the equation (A.1) does not depend on the choice of $\gamma_{z,w}$.

Let $\|\cdot\|'_t$ denote the norm (A.1) on the surface M_t. Then, the analogue of Theorem A.1 holds, for $\|\cdot\|'_t$. This assertion is essentially Lemma 4.4 from [AthF]. For a self-contained proof in this notation see [EMR, §7].)

The Avila-Gouëzel-Yoccoz (AGY) norm. The Hodge norm on relative cohomology behaves badly in the thin part of Teichmüller space. Therefore, we will use instead the Avila-Gouëzel-Yoccoz norm $\|\cdot\|_Y$ defined in [AGY], some properties of which were further developed in [AG]. The norms $\|\cdot\|_Y$ and $\|\cdot\|'$ are equivalent on compact subsets of the strata $\mathcal{H}_1(\alpha)$, and therefore the decay estimates on $\|\cdot\|'$ in the style of Theorem A.1 also apply to the Avila-Gouëzel-Yoccoz norm. Furthermore, we have the following:

Theorem A.2. *Suppose $S = (M,\omega) \in \mathcal{H}(\alpha)$. Let $\|\cdot\|_t$ denote the Avila-Gouëzel-Yoccoz (AGY) norm on the surface $M_t = \pi(g_t S)$. Then,*

(a) *For all $\lambda \in H^1(M,\Sigma,\mathbb{R})$ and all $t > 0$,*

$$\|\lambda\|_t \leq e^t \|\lambda\|_0.$$

(b) *Suppose for some compact subset $\mathcal{K}$ of $\mathcal{M}_g$, the geodesic segment $[S, g_t S]$ spends at least half the time in $\pi^{-1}(\mathcal{K})$. Suppose $\lambda \in H^1(M, \Sigma, \mathbb{R})$ with $p(\lambda)$ orthogonal to ω. Then we have*

$$\|\lambda\|_t \le C e^{(1-\alpha)t} \|\lambda\|_0,$$

where $\alpha > 0$ depends only on $\mathcal{K}$.

A.2. **The Kontsevich-Zorich cocycle.** In the sequel, a subbundle L of the Hodge bundle is called *isometric* if the action of the Kontsevich-Zorich cocycle restricted to L is by isometries in the Hodge metric. We say that a subbundle is *isotropic* if the symplectic form vanishes identically on the sections, and *symplectic* if the symplectic form is non-degenerate on the sections. A subbundle is *irreducible* if it cannot be decomposed as a direct sum, and *strongly irreducible* if it cannot be decomposed as a direct sum on any finite cover of X.

Theorem A.3. *Let ν be a P-invariant measure, and suppose L is a P-invariant ν-measurable subbundle of the Hodge bundle. Let $\lambda_1, \dots, \lambda_n$ be the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L. Then,*

$$\sum_{i=1}^{n} \lambda_i \ge 0.$$

Proof. Let the symplectic complement $L^\dagger$ of L be defined by

$$L^\dagger(x) = \{v \ : \ v \wedge u = 0 \quad \text{for all } u \in L(x)\}. \tag{A.2}$$

Then, $L^\dagger$ is an P-invariant subbundle, and we have the short exact sequence

$$0 \to L \cap L^\dagger \to L \to L/(L \cap L^\dagger) \to 0.$$

The bundle $L/(L \cap L^\dagger)$ admits an invariant non-degenerate symplectic form, and therefore, the sum of the Lypunov exponents on $L/(L \cap L^\dagger)$ is 0. Therefore, it is enough to show that the sum of the Lyapunov exponents on the isotropic subspace $L \cap L^\dagger$ is 0. Thus, without loss of generality, we may assume that L is isotropic.

Let $\{c_1, \dots, c_n\}$ be a Hodge-orthonormal basis for the bundle L at the point $S = (M, \omega)$, where M is a Riemann surface and ω is a holomorphic 1-form on M. For $g \in SL(2, \mathbb{R})$, let $V_S(g)$ denote the Hodge norm of the polyvector $c_1 \wedge \cdots \wedge c_n$ at the point gS, where the vectors c_i are transported using the Gauss-Manin connection. Since $V_S(kg) = V_S(g)$ for $k \in SO(2)$, we can think of V_S as a function on the upper half plane $\mathbb{H}$. From the definition of V_S and the multiplicative ergodic theorem, we see that for ν-almost all $S \in X$,

$$\lim_{t \to \infty} \frac{\log V_S(g_{-t})}{t} = -\sum_{i=1}^{n} \lambda_i, \tag{A.3}$$

where the λ_i are as in the statement of Theorem A.3.

Let Δ_{hyp} denote the hyperbolic Laplacian operator (along the Teichmüller disk). By [FoMZ, Lemma 2.8] (see also [Fo, Lemma 5.2 and Lemma 5.2']) there exists a non-negative function $\Phi : X \to \mathbb{R}$ such that for all $S \in X$ and all $g \in SL(2,\mathbb{R})$,

$$(\Delta_{hyp} \log V_S)(g) = \Phi(gS).$$

We now claim that the Kontsevich-Zorich type formula

$$\sum_{i=1}^{n} \lambda_i = \int_X \Phi(S)\, d\nu(S) \tag{A.4}$$

holds, which clearly implies the theorem. The formula (A.4) is proved in [FoMZ] (and for the case of the entire stratum in [Fo]) under the assumption that the measure ν is invariant under $SL(2,\mathbb{R})$. However, in the proofs, only averages over "large circles" in $\mathbb{H} = SO(2)\backslash SL(2,\mathbb{R})$ are used. Below we show that a slightly modified version of the proof works under the a-priori weaker assumption that ν is invariant under $P = AN \subset SL(2,\mathbb{R})$. This is not at all surprising, since large circles in $\mathbb{H}$ are approximately horocircles (i.e. orbits of N).

We now begin the proof of (A.4), following the proof of [FoMZ, Theorem 1].

Since (A.3) holds for ν-almost all S and ν is N-invariant, (A.3) also holds for holds for almost all $S_0 \in X$ and almost all $S \in \Omega_N S_0$, where

$$\Omega_N = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \;:\; |s| \le 1 \right\} \subset N.$$

We identify $SO(2)\backslash \mathrm{SL}(2,\mathbb{R})S_0$ with $\mathbb{H}$ so that S_0 corresponds to i. Then $\Omega_N S_0$ corresponds to the line horizontal line segment connecting $-1+i$ to $1+i$. Let $\epsilon = e^{-2t}$. Then, $g_{-t}\Omega_N S_0$ corresponds to the line segment connecting $-1+i\epsilon$ to $1+i\epsilon$.

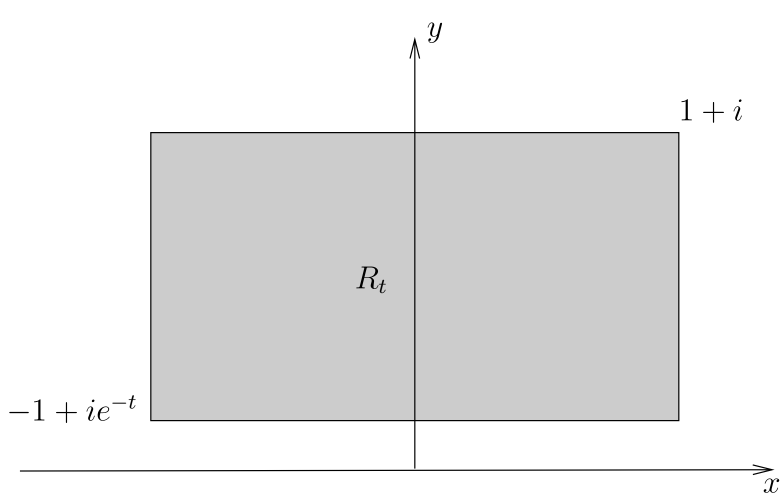

Figure 7. Proof of Theorem A.3.

Let $f(z) = \log V_{S_0}(SO(2)z)$. Note that $\nabla_{hyp} f$ is bounded (where ∇_{hyp} is the gradient with respect to the hyperbolic metric on $\mathbb{H}$). Then, (A.3) implies that for almost all $x \in [-1,1]$,

$$-\sum_{i=1}^{n} \lambda_i = \lim_{T\to\infty} \frac{f(x+ie^{-T}) - f(x+i)}{T} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\partial}{\partial t}\left[f(x+ie^{-t})\right]\, dt$$

Integrating the above formula from $x = -1$ to $x = 1$, we get (using the bounded convergence theorem),

$$-\sum_{i=1}^{n} \lambda_i = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_{-1}^{1} \frac{\partial}{\partial t}\left[f(x+ie^{-t})\right]\, dx\right) dt$$

Let R_t denote the rectangle with corners at $-1+ie^{-t}$, $1+ie^{-t}$, $1+i$ and $-1+i$, see Figure 7. We now claim that

$$\int_{-1}^{1} \frac{\partial}{\partial t}\left[f(x+ie^{-t})\right]\, dx = e^{-2t}\int_{\partial R_t} \frac{\partial f}{\partial n} + O(te^{-2t}), \tag{A.5}$$

where $\frac{\partial f}{\partial n}$ denotes the normal derivative of f with respect to the hyperbolic metric. Indeed, the integral over the bottom edge of the rectangle R_t the left hand side of (A.5) coincides with the right hand side of (A.5) (the factor of e^{-2t} appears because the hyperbolic length element is $dx/y^2 = e^{-2t}\, dx$.) The partial derivative $\frac{\partial f}{\partial n}$ is uniformly bounded, and the hyperbolic lengths of the other three sides of ∂R_t are $O(t)$. Therefore (A.5) follows.

Now, by Green's formula (in the hyperbolic metric),

$$-\int_{\partial R_t} \frac{\partial f}{\partial n} = \int_{R_t} \Delta_{hyp} f = \int_{R_t} \Phi,$$

We get, for almost all S_0,

$$\sum_{i=1}^{n} \lambda_i = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(e^{-2t}\int_{R_t} \Phi\right) dt \geq 0.$$

This completes the proof of the Theorem. It is also easy to conclude (by integrating over S_0) that (A.4) holds. $\square$

Theorem A.4. *Let ν be an $SL(2,\mathbb{R})$-invariant measure, and suppose L is an $SL(2,\mathbb{R})$-invariant ν-measurable subbundle of the Hodge bundle. Suppose all the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L vanish. Then, the action of the Kontsevich-Zorich cocycle on L is isometric with respect to the Hodge inner product, and the orthogonal complement $L^\perp$ of L with respect to the Hodge inner product is also an $SL(2,\mathbb{R})$-invariant subbundle.*

Proof. The first assertion is the content of [FoMZ, Theorem 3]. The second assertion then follows from [FoMZ, Lemma 4.3]. $\square$

Theorem A.5. *Let ν be an $SL(2,\mathbb{R})$-invariant measure, and suppose L is an $SL(2,\mathbb{R})$-invariant ν-measurable subbundle of the Hodge bundle. Suppose L is isotropic. Then all the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L vanish (and thus Theorem A.4 applies to L).*

Proof. For a point $x \in X$ and an isotropic k-dimensional subspace I_k, let $\Phi_k(x, I_k)$ be as in [FoMZ, (2.46)] (or [Fo, Lemma 5.2']). We have from [FoMZ, Lemma 2.8] that

$$\Phi_k(x, I_k) \leq \Phi_j(x, I_j) \quad \text{if } i < j \text{ and } I_k \subset I_j.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to L. Let $V_j(x)$ denote the direct sum of all the Lyapunov subspaces corresponding to exponents $\lambda_i \geq \lambda_j$. By definition, $V_n(x) = L(x)$. Suppose $j = n$ or $\lambda_j \neq \lambda_{j+1}$. Then, by [FoMZ, Corollary 3.1] the following formula holds:

$$\lambda_1 + \cdots + \lambda_j = \int_X \Phi_j(x, V_j(x))\, d\nu(x)$$

(This formula is proved in [Fo] for the case where ν is Lebesgue measure and L is the entire Hodge bundle).

We will first show that all the λ_j have the same sign. Suppose not, then we must have $\lambda_n < 0$ but not all $\lambda_j < 0$. Let k be maximal such that $\lambda_k \neq \lambda_n$. Then

$$\lambda_1 + \cdots + \lambda_k = \int_X \Phi_k(x, V_k(x))\, d\nu(x)$$

and

$$\lambda_1 + \cdots + \lambda_n = \int_X \Phi_n(x, L(x))\, d\nu(x)$$

But $\Phi_k(x, V_k(x)) \leq \Phi_n(x, L(x))$ since $V_k(x) \subset L(x)$. Thus,

$$\lambda_{k+1} + \cdots + \lambda_n \geq 0. \tag{A.6}$$

But by the choice of k, all the terms in (A.6) are equal to each other. This implies that $\lambda_n \geq 0$, contradicting our assumption that $\lambda_n < 0$. Thus all the λ_j, $1 \leq j \leq n$ have the same sign. Since ν is assumed to be $SL(2,\mathbb{R})$-invariant, and any diagonalizable $g \in SL(2,\mathbb{R})$ is conjugate to its inverse, we see that e.g. the λ_j cannot all be positive. Hence, all the Lyapunov exponents λ_j are 0. $\square$

Algebraic Hulls. The algebraic hull of a cocycle is defined in [Zi2]. We quickly recap the definition: Suppose H is an an $\mathbb{R}$-algebraic group, and let $A : G \times X \to H$ be a cocycle. We say that the $\mathbb{R}$-algebraic subgroup H' of H is the *algebraic hull* of

A if H' is the smallest $\mathbb{R}$-algebraic subgroup of H such that there exists a measurable map $C : X \to H$ such that

$$C(gx)^{-1}A(g,x)C(x) \in H' \qquad \text{for almost all } g \in G \text{ and almost all } x \in X.$$

It is shown in [Zi2] (see also [MZ, Theorem 3.8]) that the algebraic hull exists and is unique up to conjugation.

Theorem A.6. *Let ν be an $SL(2,\mathbb{R})$-invariant measure. Then,*

(a) *The ν-algebraic hull $\mathcal{G}$ of the Kontsevich-Zorich cocycle is semisimple.*
(b) *On any finite cover of X, each ν-measurable irreducible subbundle of the Hodge bundle is either symplectic or isotropic.*

Remark. The fact that the algebraic hull is semisimple for $SL(2,\mathbb{R})$-invariant measures is key to our approach.

Proof. Suppose L is an invariant subbundle. It is enough to show that there exists an invariant complement to L. Let the symplectic complement $L^\dagger$ of L be defined as in (A.2). Then, $L^\dagger$ is also an $SL(2,\mathbb{R})$-invariant subbundle, and $K = L \cap L^\dagger$ is isotropic. By Theorem A.5, K is isometric, and $K^\perp$ is also $SL(2,\mathbb{R})$-invariant. Then,

$$L = K \oplus (L \cap K^\perp), \qquad L^\dagger = K \oplus (L^\dagger \cap K^\perp),$$

and

$$H^1(M,\mathbb{R}) = K \oplus (L \cap K^\perp) \oplus (L^\dagger \cap K^\perp)$$

Thus, $L^\dagger \cap K^\perp$ is an $SL(2,\mathbb{R})$-invariant complement to L. This proves (a). Iterating this procedure, one obtains (b), and the same could be done on any finite cover. $\square$

The Forni subspace.

Definition A.7 (Forni Subspace)**.** Let

$$F(x) = \bigcap_{g \in SL(2,\mathbb{R})} g^{-1}(\operatorname{Ann} B^{\mathbb{R}}_{gx}), \tag{A.7}$$

where for $\omega \in X$ the quadratic form $B^{\mathbb{R}}_\omega(\cdot,\cdot)$ is as defined in [FoMZ, (2.33)].

Remark. It is clear from the definition, that as long as its dimension remains constant, $F(x)$ varies real-analytically with x.

Theorem A.8. *Suppose ν is an $SL(2,\mathbb{R})$-invariant measure. Then the subspaces $F(x)$ where x varies over the support of ν form the maximal ν-measurable $SL(2,\mathbb{R})$-invariant isometric subbundle of the Hodge bundle.*

Proof. Let $F(x)$ be as defined in (A.7). Then, F is an $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle, and the restriction of $B^{\mathbb{R}}_x$ to $F(x)$ is identically 0. Then, by [FoMZ, Lemma 1.9], F is isometric.

Now suppose M is any other ν-measurable isometric $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle. Then by [FoMZ, Theorem 2], $M(x) \subset \operatorname{Ann} B_x^{\mathbb{R}}$. Since M is $SL(2,\mathbb{R})$-invariant, we have $M \subset F$. Thus F is maximal. □

Theorem A.9. *On any finite cover of X,*

(a) *The Forni subspace is symplectic, and its symplectic complement $F^\dagger$ coincides with its Hodge complement $F^\perp$.*
(b) *Any invariant subbundle of $F^\perp$ is symplectic, and the restriction of the Kontsevich-Zorich cocycle to any invariant subbundle of $F^\perp$ has at least one non-zero Lyapunov exponent.*

Proof. Suppose the subspace $F^\perp$ is not symplectic. Let let $L = F^\perp \cap (F^\perp)^\dagger$. Then L is isotropic, and therefore by Theorem A.5 and Theorem A.4, L is an $SL(2,\mathbb{R})$-invariant isometric subspace. But then we would have $L \subset F$ which contradicts the maximality of F (Theorem A.8). Therefore $F^\perp$ is symplectic.

By Theorem A.6, we may pass to finite cover on which both F and $F^\perp$ decompose into a direct sum of strongly irreducible subbundles. Let M be an strongly irreducible subbundle of $F^\perp$. Then, in view of Theorem A.4 and the maximality of F, M must have at least one non-zero Lyapunov exponent. In particular, in view of Theorem A.5, M cannot be isotropic, so it must be symplectic in view of Theorem A.6 (b). This proves the statement (b).

Since $F^\perp$ is symplectic, $(F^\perp)^\dagger$ is $SL(2,\mathbb{R})$-invariant and complementary to $F^\perp$. Note that F is also $SL(2,\mathbb{R})$-invariant and complementary to $F^\perp$. In order to conclude that $(F^\perp)^\dagger = F$, it enough to show that there is a unique $SL(2,\mathbb{R})$-invariant complement to $F^\perp$. For this, it is enough to show that for any strongly irreducible subbundles $M_1 \subset F^\perp$ and $M_2 \subset F$, the algebraic hulls $\mathcal{G}(M_i)$ of the restriction of the Kontsevich-Zorich cocycle to M_i are not isomorphic to each other. But the later statement is clear, since $\mathcal{G}(M_2)$ is compact and $\mathcal{G}(M_1)$ is not (since it has at least one non-zero Lyapunov exponent by (b)). Thus, $(F^\perp)^\dagger = F$. Since we already showed that $F^\perp$ is symplectic, this implies that so is F, which completes the proof of (a). □

B. Entropy and the Teichmüller geodesic flow

The contents of this section are well-known, see e.g. [LY], [MaT] and also [BG]. However, for technical reasons, the statements we need do not formally follow from the results of any of the above papers. Our setting is intermediate between the homogeneous dynamics setting of [MaT] and the general C^2-diffeomorphism on a compact manifold setup of [LY], but it is closer to the former than the latter. What follows is a lightly edited but almost verbatim reproduction of [MaT, §9], adapted to the setting of Teichmüller space. It is included here primarily for the convenience of the reader. The (minor) differences between our presentation and that of [MaT] are

related to the lack of uniform hyperbolicity outside of compact subsets of the space, and some notational changes due to the fact that our space is not homogeneous.

Notation. Let g_t denote the Teichmüller geodesic flow. In this section, ν is a g_t-invariant probability measure on X. Let $d(,)$ denote the Hodge distance on X. Fix a point $p \in X$ such that every neighborhood of p in X has positive ν-measure. Fix relatively compact neighborhoods B' and C' of 0 in W^- and W^{0+} respectively. Let

$$B = \{p+v \ : \ v \in B'\}, \quad C = \{p+w \ : \ w \in C'\}, \quad D = \{p+v+w \ : \ v \in B', w \in C'\}$$

We assume that B' and C' are sufficiently small so that D is contractible. For $c \in C$, let $B'[c] = \{c + v \ : \ v \in B'\}$.

Lemma B.1. *(cf.* [MaT, Lemma 9.1]*) There exists $s > 0$, $C_1 \subset C$ and for each $c \in C_1$ there exists a subset $E[c] \subset W^-[c]$ such that*

(1) *$E[c] \subset B'[c]$.*
(2) *$E[c]$ is open in $W^-[c]$, and the subset $E \equiv \bigcup_{c \in C_1} E[c]$ satisfies $\nu(E) > 0$.*
(3) *Let $T = g_s$ denote the time s map of the geodesic flow. Then whenever*

$$T^n E[c] \cap E \neq 0, \qquad c \in C_1, \qquad n > 0,$$

we have $T^n E[c] \subset E$.

Proof. Fix a compact subset $K_1 \subset X$, with $\nu(K_1^c) < 0.01$. Then by the Birkhoff ergodic theorem, for every $\delta > 0$ there exists $R > 0$ and a subset E_1 with $\nu(E_1) > 1-\delta$ such that for all $x \in E$ and all $N > R$,

$$|\{n \in [1, N] \ : \ g_n x \in K_1\}| \geq (1/2)N.$$

By choosing $\delta > 0$ small enough, we may assume that $\nu(D \cap E_1) > 0$. Let

$$C_1 = \{c \in C \ : \ c + v \in D \cap E_1 \text{ for some } v \in W^-(c) \ \}.$$

Then there exists a compact $K \supset K_1$ such that for all $c \in C_1$ and all $x \in B'(c)$,

$$|\{n \in [1, N] \ : \ g_n x \in K\}| \geq (1/2)N.$$

By Lemma 3.4 there exists $\alpha > 0$ such that for all $c \in C_1$ and all $x \in B^-(c)$,

$$d(g_n x, g_n c) \leq \begin{cases} d(x, c) & \text{if } n \leq R \\ d(x,c)e^{-\alpha(n-R)} & \text{if } n > R \end{cases} \tag{B.1}$$

Therefore we may choose $s > 0$ such that if we let $T = g_s$ denote the time s map of the geodesic flow, then for all $c \in C_1$ and all $x \in B^-(c)$,

$$d(Tx, Ty) \leq \frac{1}{10} d(x, y). \tag{B.2}$$

We can assume that there exists $a > 0$ so that for all $c \in C$, $B'[c]$ contains the intersection with $W^-[c]$ of a sphere in the Hodge metric of radius $a/2$ and centered at c. Let

$$a_0 = \frac{a}{10} \tag{B.3}$$

Let $B'_0[c] \subset W^-[c]$ denote the sphere in the Hodge metric of radius a_0 and centered at c. Let $E^{(0)}[c] = B'_0[c]$, and for $j > 0$ let

$$E^{(j)}[c] = E^{(j-1)}[c] \cup \{T^n B'_0[c'] \;:\; c' \in C_1, n > 0 \text{ and } T^n B'_0[c'] \cap E^{(j-1)}[c] \neq 0\}.$$

Let

$$E[c] = \bigcup_{j \geq 0} E^{(j)}[c], \qquad \text{and } E = \bigcup_{c \in C_1} E[c].$$

It easily follows from the above definition that $E[c]$ has the properties (2) and (3). To show (1), it is enough to show that for each j,

$$d(x, c) < a/2, \quad \text{for all } x \in E^{(j)}[c]. \tag{B.4}$$

This is done by induction on j. The case $j = 0$ holds since $a_0 = a/10 < a/2$. Suppose (B.4) holds for $j - 1$, and suppose $x \in E^{(j)}[c] \setminus E^{(j-1)}[c]$. Then there exist $c_0 = c, c_1, \ldots, c_j = x$ in C_1 and non-negative integers $n_0 = 0, \ldots, n_j$ such that for all $1 \leq k \leq n$,

$$T^{n_k}(B'_0[c_k]) \cap T^{n_{k-1}}(B'_0[c_{k-1}]) \neq \emptyset. \tag{B.5}$$

Let $1 \leq k \leq j$ be such that n_k is minimal. Recall that $B'[y] \cap B'[z] = \emptyset$ if $y \neq z$, $y \in C_1$, $z \in C_1$. Therefore, in view of the inductive assumption, $n_k \geq 1$. Applying T^{-n_k} to (B.5) we get

$$\left(\bigcup_{i=1}^{k-1} T^{n_i - n_k} B'_0[c_i] \right) \cap B'_0[c_k] \neq \emptyset, \quad \text{and} \quad \left(\bigcup_{i=k+1}^{j} T^{n_i - n_k} B'_0[c_i] \right) \cap B'_0[c_k] \neq \emptyset.$$

Therefore, in view of (B.5), and the definition of the sets $E^{(j)}[c]$,

$$\left(\bigcup_{i=1}^{k} T^{n_i - n_k} B'_0[c_i] \right) \subset E^{(k)}[c_k], \quad \text{and} \quad \left(\bigcup_{i=k}^{j} T^{n_i - n_k} B'_0[c_i] \right) \subset E^{(j-k)}[c_k]$$

By the induction hypothesis, $\operatorname{diam}(E^{(k)}[c_k]) \leq a/2$, and $\operatorname{diam}(E^{(j-k)}[c_k]) < a/2$. Therefore,

$$\operatorname{diam} \left(\bigcup_{i=1}^{j} T^{n_i - n_k} B'_0[c_i] \right) \leq a.$$

Then, applying T^{n_k} we get,

$$\operatorname{diam} \left(\bigcup_{i=1}^{j} T^{n_i} B'_0[c_i] \right) \leq \frac{a}{10}$$

Since $\operatorname{diam}(B_0'[c]) \le a/10$, we get

$$\operatorname{diam}\left(\bigcup_{i=0}^{j} T^{n_i} B_0'[c_i]\right) \le \operatorname{diam}(B_0'[c_0]) + \operatorname{diam}(\left(\bigcup_{i=0}^{j} T^{n_i} B_0'[c_i]\right) \le \frac{a}{10} + \frac{a}{10} \le \frac{a}{2}.$$

But the set on the left-hand-side of the above equation contains both $c = c_0$ and $x = c_j$. Therefore $d(c, x) \le a/2$, proving (B.4). Thus (1) holds. $\square$

Lemma B.2. *(Mane) Let E be a measurable subset of X, with $\nu(E) > 0$. If ν is a measure on E and $q : E \to (0, 1)$ is such that $\log q$ is ν-integrable, then there exists a countable partition $\mathcal{P}$ of E with entropy $H(\mathcal{P}) < \infty$ such that, if $\mathcal{P}(x)$ denotes the atom of $\mathcal{P}$ containing x, then $\operatorname{diam}\mathcal{P}(x) < q(x)$.*

Proof. See [M1] or [M2, Lemma 13.3] $\square$

Definition B.3. We say that a measurable partition ξ of the measure space (X, ν) is *subordinate* to a system of subspaces $V(x) \subset H^1(\cdot, \cdot)$ if for almost all (with respect to ν) $x \in X$, we have

(a) $\xi[x] \subset V[x]$ where $\xi[x]$ denotes, as usual, the element of ξ containing x.
(b) $\xi[x]$ is relatively compact in $V[x]$.
(c) $\xi[x]$ contains a neighborhood of x in $V[x]$.

Let η and η' be measurable partitions of (X, ν). We write $\eta \le \eta'$ if $\eta[x] \supset \eta'[x]$ for for almost all (with respect to ν) $x \in X$. We define a partition $T\eta$ by $(T\eta)[x] = T(\eta[T^{-1}(x)])$.

Proposition B.4. *Assume that μ is T-ergodic. Then there exists a measurable partition η of the measure space (X, ν) with the following properties:*

(i) *η is subordinate to W^-.*
(ii) *η is T-invariant, i.e. $\eta \le T\eta$.*
(iii) *The mean conditional entropy $H(T\eta \mid \eta)$ is equal to the entropy $h(T, \nu)$ of the automorphism $x \to Tx$ of the measure space (X, ν).*

Proof. Let $E[c]$ and E be as in Lemma B.1. Denote by $\pi : E \to C_1$ the natural projection ($\pi(x) = c$ if $x \in E[c]$). We set $\eta[x] = E(\pi(x))$ for every $x \in E$.

We claim that it is enough to find a countable measurable partition ξ of (X, ν) such that $H(\xi) < \infty$ and $\eta[x] = \xi^-[x]$ for almost all $x \in E$ where $\xi^- = \bigvee_{n=0}^{\infty} T^{-n}\xi$ is the product of the partitions $T^{-n}\xi$, $0 \le n < \infty$.

Indeed, suppose the claim holds. Then it is clear that η is T-invariant. The set of $x \in X$ for which properties (a) and (b) (resp. (c)) in the definition of a subordinate partition are satisfied is T^{-1}-invariant (resp. T-invariant) and contains E. But $\nu(E) > 0$ and ν is T-ergodic. Therefore, η is subordinate to W^-. To check the property (iii) it is enough to show that the partition $\xi_s = \bigvee_{k=-\infty}^{\infty} T^k\xi$ is the partition into points, see [R, §9], or [KH, §4.3]. By [Fo] or [ABEM, Theorem 8.12] $\xi_s(x) = \{x\}$

if $T^{-n}x \in E$ for infinitely many n. (Recall that by the construction of E, any such geodesic will spend at least half the time in the compact set K). But $\nu(E) > 0$ and ν is T-ergodic. Hence $\xi_s[x] = \{x\}$ for almost all x, which completes the proof of the claim.

Let us construct the desired partition ξ. For $x \in E$, let $n(x)$ be the smallest positive integer n such that $T^n x \in E$. Let

$$F_m = \bigcup_{\{x \,:\, n(x)=m\}} \quad \bigcup_{0 \le k \le m-1} \{T^k x\}.$$

Since $\nu(E) > 0$ and ν is T-invariant and T-ergodic, we get that

$$\bigsqcup_{m=1}^{\infty} F_m = X,$$

and therefore

$$\int_E n(x)\, d\nu(x) = \sum_{m=1}^{\infty} |F_m| = 1. \tag{B.6}$$

Define a probability measure ν' on C_1 by

$$\nu'(F) = \frac{\nu(\pi^{-1}(F))}{\nu(E)}, \qquad F \subset C_1. \tag{B.7}$$

Property (3) of the family $\{E[c] \;:\; c \in C_1\}$ implies that $n(x)$ is constant on every $E[c]$, $c \in C_1$. Therefore, in view of (B.6) and (B.7),

$$\int_{C_1} n(c)\, d\nu'(c) < \infty. \tag{B.8}$$

By Lemma 3.5, there exists $\kappa > 1$ such that for all x, $y \in X$,

$$d(Tx, Ty) \le \kappa d(x, y).$$

Since the function $n(c)$ is ν'-integrable, one can find a positive function $q(c) < \kappa^{-2n(c)}$, $c \in C_1$ such that $\log q$ is ν'-integrable, and the ν'-essential infimum $\operatorname{ess\,inf}_{c \in C_1} q(c)$ is 0.

After replacing, if necessary, B' and C' by smaller subsets we can find $\epsilon > 0$ such that

(a) $d(x, y) < 2d(\pi(x), \pi(y))$ whenever $x, y \in E$ and $d(x, y) < \epsilon$, and
(b) if $x, y \in C_1$ then $d(x, y) < \epsilon$.

Since the function $\log q(c)$ is ν'-integrable, there exists a countable measurable partition $\mathcal{P}$ of C_1 such that $H(\mathcal{P}) < \infty$ and $\operatorname{diam} \mathcal{P}(x) < \frac{\epsilon}{2} q(x)$ for almost all $x \in C_1$ (see Lemma B.2). Now we define a countable measurable partition ξ of X by

$$\xi(x) = \begin{cases} \pi^{-1}(\mathcal{P}(\pi(x))) & \text{if } x \in E \\ X \setminus E & \text{if } x \notin E. \end{cases}$$

Since $H(\mathcal{P}) < \infty$ we get using (B.7) that $H(\xi) < \infty$. It remains to show that $\xi^-[x] = \eta[x]$ for almost all $x \in E$. It follows from the property (3) of the family $\{E[c]\}$ that $\eta[z] \subset \xi^-[z]$. Let x and y be elements in E with $\xi^-[x] = \xi^-[y]$. Since $\eta[z] \subset \xi[z]$, we can assume that $x, y \in C_1$. Then $d(x, y) < \epsilon$. Set $x_1 = x$, $y_1 = y$ and define by induction

$$x_{k+1} = T^{n(x_k)} x_k, \quad y_{k+1} = T^{n(y_k)} y_k.$$

Then, the sequence $\{x_k\}_{k\in\mathbb{N}}$ (resp. $\{y_k\}_{k\in\mathbb{N}}$) is the part of the T-orbit of x (resp. T-orbit of y) which lies in E.

Let $\tilde{x_1}$, $\tilde{y_1}$ be the lifts of $x_1 = x$ and $y_1 = y$ to Teichmüller space, and let $\tilde{x}_k$, $\tilde{y}_k$ be defined inductively by

$$\tilde{x}_{k+1} = T^{n(x_k)} \tilde{x}_k, \quad \tilde{y}_{k+1} = T^{n(y_k)} \tilde{y}_k.$$

Then $\tilde{x}_k$ and $\tilde{y}_k$ are lifts of x_k and y_k respectively. We now claim that for all $k \geq 0$,

$$d(\tilde{x}_k, \tilde{y}_k) < \epsilon q(\pi(x_k)). \tag{B.9}$$

If $k = 1$, the inequality (B.9) is true because $\operatorname{diam} \mathcal{P}(x) < \frac{\epsilon}{2} q(\pi(x))$ and $\mathcal{P}(x) = \mathcal{P}(y)$. Assume that (B.9) is proved for k. Then

$$d(\tilde{x}_{k+1}, \tilde{y}_{k+1}) = d(T^{n(x_k)}\tilde{x}_k, T^{n(x_k)}\tilde{y}_k) \leq \kappa^{n(x_k)} d(\tilde{x}_k, \tilde{y}_k) \leq \kappa^{n(x_k)} \epsilon q(\pi(x_k)) \leq \epsilon.$$

Then since x_{k+1} and y_{k+1} belong to the same element of the partition ξ (because $\xi^-[x] = \xi^-[y]$) and $\operatorname{diam}(\mathcal{P}(x_k)) \leq \frac{\epsilon}{2} q\pi(x_k)$, we get from condition (b) in the definition of $\epsilon > 0$ that (B.9) is true for $k + 1$.

Since the measure ν is T-ergodic and $\operatorname{ess\,inf} q(c) = 0$ we may assume that $\liminf_{k\to\infty} q(\pi(x_k)) = 0$ (since this holds for almost all $x \in E$). Then (B.9) implies that

$$\liminf_{k\to\infty} d(\tilde{x}_k, \tilde{y}_k) = 0.$$

By the definition of $\tilde{x}_k$, $\tilde{y}_k$, there exists a sequence $m_k \to +\infty$ such that $\tilde{x}_k = T^{m_k}\tilde{x}$, $\tilde{y}_k = T^{m_k}\tilde{y}$. Thus,

$$d(T^{m_k}\tilde{x}, T^{m_k}\tilde{y}) = 0.$$

But, by construction $\tilde{x}$ and $\tilde{y}$ are on the same leaf of W^{0+}. This contradicts the non-contraction property of the Hodge distance [ABEM, Theorem 8.2], unless $\tilde{x} = \tilde{y}$. Thus we must have $x = y$. $\square$

Lemma B.5. *(see* [LS, Proposition 2.2]*.) Let T be an automorphism of a measure space (X, ν), $\nu(X) < \infty$, and let f be a positive finite measurable function defined on X such that*

$$\log^- \frac{f \circ T}{f} \in L^1(X, \nu), \quad \textit{where } \log^-(a) = \min(\log a, 0).$$

Then

$$\int_X \log \frac{f \circ T}{f} \, d\nu = 0.$$

Proposition B.6. *Let $V^-(x) \subset W^-(x)$ be an T-equivariant family of subspaces. Suppose there exists a T-invariant measurable partition η of (X, ν) subordinate to V^-. Then the following hold:*

(i) *If the conditional measures of ν along V^- are Lebesgue, then*

$$H(T\eta \mid \eta) = s\Delta(V^-)$$

where $H(T\eta \mid \eta)$ is the mean conditional entropy, and

$$\Delta(V^-) = \sum_{i \in I(V)} (1 - \lambda_i),$$

where $I(V)$ are the Lyapunov subspaces in V (counted with multiplicity), and λ_i are the corresponding Lyapunov exponents of the Kontsevich-Zorich cocycle.

(ii) *$H(T\eta \mid \eta) \leq s\Delta(V^-)$. The equality $H(T\eta \mid \eta) = s\Delta(V^-)$ implies that the conditional measures of ν along V^- are Lebesgue.*

Proof. Since $\eta \leq T\eta$ for almost all $x \in X$ we have a partition η_x of $\eta[x]$ such that $\eta_x[y] = (T\eta)[y]$ for almost all $y \in \eta[x]$. Denote by τ the Lebesgue measure on V^-. (Here we pick some normalization of the Lebesgue measure on the connected components of the intersections of the leaves of V^- with a fixed fundamental domain). Since $\eta[x] \subset V^-[x]$, τ induces a measure on $\eta[x]$ which we will denote also by τ. Let $J(x)$ denote the Jacobian of the restriction of the map T to $V^-[x]$ at x (with respect to the Lebesgue measures τ on $V^-[x]$ and $V^-[Tx]$). Then, by the Osceledec multiplicative ergodic theorem, for almost all $x \in X$,

$$-s\Delta(V^-) = \lim_{N\to\infty} \frac{1}{N} \log \frac{d(T^{-N}\tau)(x)}{d\tau(x)} = -\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log J(T^{-n}x).$$

Integrating both sides over X, we get

$$-\int_X \log J(x)\, d\nu(x) = s\Delta(V^-).$$

Put $L(x) = \tau(\eta[x])$ and $\tau_x = \tau/L(x)$, $x \in X$. Note that on $\eta[x]$ we have a conditional probability measure ν_x induced by ν. Put $p(x) = \tau_x(\eta_x[x])$ and $r(x) = \nu_x(\eta_x[x])$. Then since $\eta_x[x] = T(\eta[T^{-1}x])$ one easily sees that $p(x) = \frac{L(T^{-1}x)}{J(x)L(x)}$.

From its definition, $p(x) \leq 1$. Therefore, in view of Lemma B.5, we obtain

$$-\int_X \log p(x)\, d\nu(x) = s\Delta(V^-). \tag{B.10}$$

Assume that the conditional measures along V^- are Lebesgue. Then, $\nu_x = \tau_x$ for almost all $x \in X$, in particular, $p(x) = r(x)$ for almost all $x \in X$. But

$$-\int_X \log r(x)\, d\nu(x) = H(T\eta \mid \eta). \tag{B.11}$$

This in view of (B.10) proves (i).

Let $Y_i(x)$, $1 \le i < \infty$ denote the elements of the countable partition η_x of $\eta[x]$. Then we have

$$\int_{\eta(x)} \log p(y)\, d\nu_x(y) - \int_{\eta(x)} \log r(y)\, d\nu_x(y) = \sum_{i=1}^{\infty} \log \frac{\tau_x(Y_i(x))}{\nu_x(Y_i(x))} \nu_x(Y_i(x)). \tag{B.12}$$

We have that

$$\sum_{i=1}^{\infty} \tau_x(Y_i(x)) \le 1, \tag{B.13}$$

and

$$\sum_{i=1}^{\infty} \nu_x(Y_i(x)) = 1. \tag{B.14}$$

(In (B.13), we can have inequality because apriori it is possible that the measure τ_x of $\eta[x] \setminus \bigcup_{i=1}^{\infty} Y_i(x)$ is positive). From (B.12), (B.13) and (B.14), using the convexity of log we get that

$$\int_{\eta(x)} \log p(y)\, d\nu_x(y) \le \int_{\eta(x)} \log r(y)\, d\nu_x(y).$$

and the equality holds if and only if $p(y) = r(y)$ i.e. $\tau_x(\eta_x[y]) = \nu_x(\eta_x[y])$ for all $y \in \eta[x]$. y q(x). Now using integration over the quotient space $(X, \nu)/\eta$ of the measure space (X, ν) by η, we get from (B.10) and (B.11) that $H(T\eta \mid \eta) \le \Delta(V^-)$ and the equality holds if and only if $\tau_x((T\eta)[x]) = \nu_x((T\eta)[x])$ for almost all $x \in X$.

Assume that $H(T\eta \mid \eta) = s\Delta(V^-)$. Then $H(T^k\eta \mid \eta) = ks\Delta(V^-)$ for every $k > 0$. Using the same argument as above and replacing T by T^k, we get that $\tau_x((T^k\eta)[x]) = \nu_x((T^k\eta)[x])$ for any $k > 0$ and almost all $x \in X$. On the other hand since η is subordinate to V^- and T is contracting on V^-, we have that $\bigvee_{k=1}^{\infty} T^k\eta$ is the partition into points. Hence the conditional measures of ν along V agree with τ. $\square$

Theorem B.7. *Let $T = g_s$ denote the time s map of the geodesic flow. Assume that T acts ergodically on (X, ν). Let $V^-(x)$ be a T-equivariant system of subspaces of $W^-(x)$, and let $\Delta(V^-)$ be as in Proposition B.6.*

(i) *If the conditional measures of ν along V are Lebesgue, then $h(T, \nu) \ge s\Delta(V^-)$.*
(ii) *Assume that there exists a subset $\Psi \subset X$ with ν-measure 1 such that $\Psi \cap W^-[x] \subset V^-[x]$ for every $x \in \Psi$. Then $h(T, \nu) \le s\Delta(V^-)$, and equality implies that the conditional measures of ν along V are Lebesgue.*

Proof. According to Proposition B.4, there exists a measurable T-invariant partition η of (X, ν), subordinate to W^-, such that $H(T\eta \mid \eta) = h(T, \nu)$. By Corollary 3.2, we may assume that the leaf $W^-[x]$ is embedded in X and contains no points from lower strata. Set $\eta'(x) = V^-[x] \cap \eta[x]$. Then η and η' coincide on Ψ, i.e. $\eta[x] \cap \Psi = \eta'[x] \cap \Psi$.

Hence $H(T\eta \mid \eta) = H(T\eta' \mid \eta')$. By Proposition B.4 (iii), $h(T,\nu) = H(T\eta \mid \eta)$. Using Proposition B.6 (ii) we obtain that $h(T,\nu) \le s\Delta(V^-)$, and the equality implies that the conditional measures of ν along V^- are Lebesgue. □

C. Semisimplicity of the Lyapunov spectrum

In this section we work with a bit more generality then we need. Let X be a space on which $SL(2,\mathbb{R})$ acts. Let μ be a compactly supported measure on $SL(2,\mathbb{R})$ and let ν be an ergodic μ-stationary measure on X. Let L be a real vector space, and suppose $A : \mathrm{SL}(2,\mathbb{R}) \times X \to SL(L)$ is a cocycle. Let $\mathcal{G}$ be the algebraic hull of the cocycle A (see §A.2 for the definition). We may assume that a basis at every point is chosen so that for all $g \in SL(2,\mathbb{R})$ and all $x \in X$, $A(g,x) \in \mathcal{G}$.

Definition C.1. We say that $A(\cdot,\cdot)$ has an invariant system of subspaces if for ν-a.e. $x \in X$ there exists a subspace $W(x) \subset H^1(M,\mathbb{R})$ such that for ν-a.e. $g \in SL(2,\mathbb{R})$ and ν-a.e. $x \in X$, $A(g,x)W(x) = W(gx)$.

Definition C.2 (Strongly Irreducible)**.** We say that A is *strongly irreducible* if on any finite cover of X there is no nontrivial proper invariant system of subspaces of L.

Let $\hat{T} : B \times X \to B \times X$ be the forward shift, with $\beta \times \nu$ as the invariant measure. We denote elements of B by the letter a (following the convention that these refer to "future" trajectiories). If we write $a = (a_1, a_2, \dots)$ then

$$\hat{T}(a,x) = (Ta, ax)$$

(and we use the letter T to denote the shift $T(a_1,a_2,\dots) = (a_2,a_3,\dots)$. By the Osceledec multiplicative ergodic theorem, for $\beta \times \nu$ almost every $(a,x) \in B \times X$ there exists a Lyapunov flag

$$\{0\} = \hat{V}_0 \subset \hat{V}_1(a,x) \subset \hat{V}_2(a,x) \subset \hat{V}_k(a,x) = L. \tag{C.1}$$

Definition C.3. The map $\hat{T} : B \times X \to B \times X$ has *semisimple Lyapunov spectrum* if the algebraic hull $\mathcal{G}$ is block-conformal, see §4.3. In other words, $\hat{T}$ has semisimple Lyapunov spectrum if all the off-diagonal blocks labelled $*$ in (4.6) are 0.

In Appendix C our aim is to prove the following general fact:

Theorem C.4. *Suppose A is strongly irreducible. Then $\hat{T}$ has semisimple Lyapunov spectrum. Furthermore, the restriction of $\hat{T}$ to the top Lyapunov subspace V_1 consists of a single conformal block, i.e. for $\beta \times \nu$ almost every (a,x) there exists an inner product $\langle\cdot,\cdot\rangle_{a,x}$ on $\hat{V}_1(a,x)$ and a function $\lambda : B \times X \to \mathbb{R}$ such that for all $u, w \in \hat{V}_1(a,x)$,*

$$\langle a_1 u, a_1 v\rangle_{(Ta,ax)} = \lambda(a_1,x)\langle u,v\rangle_{a,x}. \tag{C.2}$$

If the algebraic hull $\mathcal{G}$ is all of $SL(L)$, then all the Lyapunov subspaces consist of a single conformal block, i.e. for all $1 \le i \le k-1$ one can define an inner product $\langle \cdot, \cdot \rangle_{a,x}$ on $\hat{V}_i(a,x)/\hat{V}_{i-1}(a,x)$ so that (C.2) holds for some function $\lambda = \lambda_i$.

The backwards shift. We will actually use the analogue of Theorem C.4 for the backwards shift. Let $T : B \times X \to B \times X$ be the (backward) shift as in §14, with β^X as defined in [BQ, Lemma 3.1] as the invariant measure. By the Osceledec multiplicative ergodic theorem, for β^X almost every $(b,x) \in B \times X$ there exists a Lyapunov flag

$$\{0\} = V_0 \subset V_1(b,x) \subset V_2(b,x) \subset V_k(b,x) = L. \tag{C.3}$$

As a corollary of Theorem C.4, we get the following:

Theorem C.5. *Suppose A is strongly irreducible. Then T has semisimple Lyapunov spectrum. Furthermore, the restriction of T to the top Lyapunov subspace V_1 consists of a single conformal block, i.e. for β^X almost every (b,x) there exists an inner product $\langle \cdot, \cdot \rangle_{b,x}$ on $V_1(b,x)$ and a function $\lambda : B \times X \to \mathbb{R}$ such that for all $u, w \in V_1(b,x)$,*

$$\langle b_0^{-1}u, b_0^{-1}v \rangle_{(Tb, b_0^{-1}x)} = \lambda(b_0, x)\langle u, v \rangle_{b,x}. \tag{C.4}$$

If the algebraic hull $\mathcal{G}$ is all of $SL(L)$, then all the Lyapunov subspaces consist of a single conformal block, i.e. for all $1 \le i \le k-1$ one can define an inner product $\langle \cdot, \cdot \rangle_{b,x}$ on $V_i(b,x)/V_{i-1}(b,x)$ so that (C.4) holds for some function $\lambda = \lambda_i$.

Remark 1. The proof of Theorem C.4 we give is essentially taken from [GM], and is originally from [GR1] and [GR2]. In our application to the Kontsevich-Zorich cocycle, the measure ν is actually μ-invariant (and not just μ-stationary). We choose to present the proof for the case of stationary measures since it is not more difficult.

We follow [GM] and present the proof of Theorem C.4 for the easier to read case where the algebraic hull $\mathcal{G}$ of the cocycle A is all of $SL(L)$. The general case of semisimple $\mathcal{G}$ is treated in [EMat].

Remark 2. It is possible to define semisimplicity of the Lyapunov spectrum in the context of the action of $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset SL(2,\mathbb{R})$ (instead of the random walk). Then the analogue of Theorem C.4 remains true; the proof would use an argument similar to the proof of Proposition 4.4. Since we will not use this statement we will omit the details.

C.1. **An Ergodic Lemma.** We recall the following well-known lemma:

Lemma C.6. *Let $T : \Omega \to \Omega$ be a transformation preserving a probability measure β. Let $F : \Omega \to \mathbb{R}$ be an L^1 function. Suppose that for β-a.e. $x \in \Omega$,*

$$\liminf \sum_{i=1}^{n} F(T^i x) = +\infty.$$

Then $\int_\Omega F \, d\beta > 0$.

Proof. This lemma is due to Atkinson [At] and Kesten [Ke]. See also [GM, Lemma 5.3], and the references quoted there. □

We will need the following variant:

Lemma C.7. *Let $T : \Omega \to \Omega$ be a transformation preserving a probability measure β. Let $F : \Omega \to \mathbb{R}$ be an L^1 function. Suppose for every $\epsilon > 0$ there exists $K_\epsilon \subset \Omega$ with $\beta(K_\epsilon) > 1 - \epsilon$ such that for β-a.e. $x \in \Omega$,*

$$\liminf\{\sum_{i=1}^{n} F(T^i x) \;\;:\;\; T^n x \in K_\epsilon\} = +\infty.$$

Then $\int_\Omega F\, d\beta > 0$.

Proof. From

$$\liminf\{\sum_{i=1}^{m} F(T^i x) | T^m \in K\} = \infty \tag{C.5}$$

we can easily show that $F, |F| \in L_1(\Omega)$. First, we can choose a subset $K \subset K_\epsilon$ with $\beta(K) > 0$, and $C > 0$ such that for all $x \in K$, we have

$$|F(x)| < C.$$

Also, without loss of generality we can assume that the action of $T : \Omega \to \Omega$ with respect to β is ergodic.

Let $A_{-1} = \{x | x \notin K\}$, $A_0 = \{x | x \in K, Tx \in K\}$, and for $n \geq 0$,

$$A_{n+1} = \{x | x \in K, Tx \notin K, ..., T^n x \notin K, T^{n+1} x \in K\}.$$

Also let $A = \cup_{n=-1}^{\infty} A_n$. Note that by ergodicity of T, for almost every $x \in \Omega$,

$$|\{i | i \geq 0, T^i(x) \in K\}| = \infty. \quad (*).$$

Define $G : \Omega \to \mathbb{R}$ defined on A (which has full measure) by

- $G(x) = 0$ if $x \in A_{-1}$, $G(x) = F(x)$ if $x \in A_0$
- $G(x) = F(x) + F(Tx) + \cdots + F(T^n(X))$ if $x \in A_{n+1}$.

Then:

(1) For almost every $x \in \Omega$ (satisfying $(C.5)$) we have

$$\lim_{n \to \infty} G(x) + G(Tx) + ... + G(T^n x) = \infty. \tag{C.6}$$

This is because

$$G(x) + G(Tx) + ... + G(T^n x) = \sum_{i=m_0}^{m-1} F(T^i x),$$

where $m_0 = \inf\{k | T^k x \in K\}$, and $m = \inf\{k | k \geq n, T^k x \in K\}$. Since for every $x \in K$, $|F(X)| < C$, $(C.5)$ implies $(C.6)$.

(2) $\int_\Omega G^+ d\beta \leq \int_\Omega F^+ d\beta < \infty$, and $\int_\Omega |G| d\beta \leq \int_\Omega |F| d\beta < \infty$.

(3) $\int_\Omega G(x)d\beta(x) \leq \int_\Omega F(x)d\beta(x)$.

In order to show 2 note that $(f+g)^+ \leq f^+ + g^+$, and $|f+g| < |f| + |g|$. Here we sketch the idea of proving 3 assuming 2. The proof of 2 is similar.

By the definition of G we can use the dominated convergence theorem, and get that

$$\int_\Omega G\, d\beta = \int_{x\in K} F(x)d\beta(x) + \sum_{i=1}^{\infty} \int_{x\in A^i} F(T^i(x))d\beta(x)$$

where $A^i = \cup_{j\geq i} A_i$. Then

$$T^i A^i = T^i K - (K \cup \cdots T^{i-1}K).$$

Also $K \cup \cup_{i=1}^{\infty} T^i A^i$ has full measure in Ω, and for $i \neq j$ $T^i A^i \cap T^j A^j$ and $K \cap T^i A^i$ have measure zero. Note that $A^i = T^{-i}(T^i(A^i))$. Since β is T invariant, we have

$$\int_{x\in A^i} F(T^i(x))d\beta(x) = \int_{x\in T^i A^i} F(x)d\beta(x),$$

and hence

$$\int_\Omega G d\beta = \int_{x\in K} F(x)d\beta(x) + \sum_{i=1}^{\infty} \int_{x\in T^i A^i} F(x)d\beta(x) = \int_\Omega F d\beta(x).$$

Now by 1, and 2, the function G satisfies the assumptions of Lemma C.6. Hence we have $\int_\Omega F d\beta = \int_\Omega G d\beta > 0$. □

C.2. **A Zero One Law.**

Lemma C.8. *Suppose h is a bounded non-negative μ-subharmonic function, i.e. for ν-almost all $x \in X$,*

$$h(x) \leq \int_G h(gx)\, d\mu(g). \tag{C.7}$$

Then h is constant ν-almost everywhere.

Proof. By the random ergodic theorem [Fu, Theorem 3.1], for ν-almost all $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_G h(gx)\, d\mu^n(g) = \int_X h\, d\nu$$

Therefore, by (C.7), for ν-almost all $x \in X$,

$$h(x) \leq \int_X h\, d\nu. \tag{C.8}$$

Let $s_0 \geq 0$ denote the essential supremum of h, i.e.

$$s_0 = \inf\{s \in \mathbb{R} \;:\; \nu(\{h > s\}) = 0\}.$$

Suppose $\epsilon > 0$ is arbitrary. We can pick $x \in X$ such that (C.8) holds and $h(x) > s_0 - \epsilon$. Then,

$$s_0 - \epsilon \leq h(x) \leq \int_X h \, d\nu \leq s_0.$$

Since $\epsilon > 0$ is arbitrary, $\int_X h \, d\nu = s_0$. Thus $h(x) = s_0$ for ν-almost all x. □

Let ν be an ergodic stationary measure on X. Fix $1 \leq s \leq n$, and let Gr_s denote the Grassmanian of s-dimensional subspaces in L. Let $\hat{X} = X \times Gr_s$. We then have an action of $SL(2,\mathbb{R})$ on $\hat{X}$, by

$$g \cdot (x, W) = (gx, A(g,x)W).$$

Let $\hat{\nu}$ be an ergodic μ-stationary measure on $\hat{X}$ which projects to ν under the natural map $\hat{X} \to X$. (Note there is always at least one such: one chooses $\hat{\nu}$ to be an extreme point among the measures which project to ν. If $\hat{\nu} = \hat{\nu}_1 + \hat{\nu}_2$ where the $\hat{\nu}_i$ are μ-stationary measures then $\nu = \pi_*(\hat{\nu}) = \pi_*(\hat{\nu}_1) + \pi_*(\hat{\nu}_2)$. Since ν is μ-ergodic, this implies that $\pi_*(\hat{\nu}_1) = \pi_*(\hat{\nu}_2) = \nu$, hence the $\hat{\nu}_i$ also project to ν. Since $\hat{\nu}$ is an extreme point among such measures, we must have $\hat{\nu}_1 = \hat{\nu}_2 = \hat{\nu}$. This $\hat{\nu}$ is μ-ergodic.)

We may write

$$d\hat{\nu}(x, U) = d\nu(x) \, d\eta_x(U),$$

where η_x is a measure on Gr_s.

Let $m = \dim(L)$. For a subspace W of L, let

$$I(W) = \{U \in Gr_s \ : \ \dim(U \cap W) > \max(0, m - \dim(U) - \dim(W))\}$$

Then $U \in I(W)$ if and only if U and W intersect more than general position subspaces of dimension $\dim(U)$ and $\dim(V)$.

Lemma C.9. *(cf.* [GM, Lemma 4.2]*)*

(i) *Suppose the cocycle is strongly irreducible on L. Then for almost all $x \in X$, and any 1-dimensional subspace $W_x \subset L$, $\eta_x(I(W_x)) = 0$.*

(ii) *Suppose the algebraic hull $\mathcal{G}$ of the cocycle is $SL(L)$. Then for almost all $x \in X$, for any nontrivial proper subspace $W_x \subset L$, $\eta_x(I(W_x)) = 0$.*

Proof. Suppose there exists a subset $E \subset X$ with $\nu(E) > 0$ and for all $x \in E$, a nontrivial subspace $W_x \subset L$ such that $\eta_x(I(W_x)) > 0$. Let $\vec{W} = (W_1, \dots, W_k)$ denote a finite collection of subspaces of L. Write

$$I(\vec{W}) = I(W_1) \cap \cdots \cap I(W_k).$$

For $x \in E$, let $\mathcal{S}_x$ denote the set of $I(\vec{W}_x)$ such that for any $\vec{W}'_x$ so that $I(\vec{W}'_x)$ is a proper subset of $I(W_x)$, we have $\nu_x(I(W'_x)) = 0$. For $\vec{W} \in S_x$, let

$$f_{I(\vec{W})}(x) = \eta_x(I(\vec{W})).$$

Since $\hat{\nu}$ is μ-stationary, we have

$$f_{I(\vec{W})}(x) = \int_G f_{I(A(g,x)\vec{W})}(gx)\, d\mu(g) \tag{C.9}$$

Let $\mathcal{S}(x) = \{I(\vec{W}) \in \mathcal{S}_x \ : \ f_{I(\vec{W})}(x) > 0\}$. Then, for $I(\vec{W}_1) \in \mathcal{S}(x)$, $I(\vec{W}_2) \in \mathcal{S}(x)$,

$$\eta_x(I(\vec{W}_1) \cap I(\vec{W}_2)) = 0.$$

Thus

$$\sum_{I(\vec{W}) \in \mathcal{S}(x)} f_{I(\vec{W})}(x) \le 1.$$

Therefore $\mathcal{S}(x)$ is at most countable. Let

$$f(x) = \max_{I(\vec{W}) \in \mathcal{S}(x)} f_{I(\vec{W})}(x). \tag{C.10}$$

Applying (C.9) to some $I(\vec{W})$ for which the max is achieved, we get

$$f(x) \le \int_G f(gx)\, d\mu(g)$$

i.e. f is a subharmonic function on X. By Lemma C.8, f is constant almost everywhere. Now substituting again into (C.9) we get that the cocycle A permutes the finite set of $I(\mathcal{W})$ where the maximum (C.10) is achieved. Therefore the same is true for the algebraic hull $\mathcal{G}$. If (ii) holds, this is a contradiction since $\mathcal{G}$ acts transitively on subspaces of L. If (i) holds then $\mathcal{G}$ must permute a finite set of subspaces of L which contradicts the strong irreducibility assumption. $\square$

C.3. **Proof of Theorem C.5.** Let L be an invariant subspace for the cocycle. (We will work with one invariant subspace at a time). Because of semisimplicity, we may assume that the L contains no nontrivial proper invariant subspaces. Let $m = \dim L$.

Definition C.10 (**(ϵ, δ)-regular).** Suppose $\epsilon > 0$ and $\delta > 0$ are fixed. A measure η on $Gr_k(L)$ is (ϵ, δ)-*regular* if for any subspace U of L,

$$\eta(Nbhd_\epsilon(I(U))) < \delta.$$

Lemma C.11. *Suppose $g_n \in GL(L)$ is a sequence of linear transformations, and η_n is a sequence of uniformly (ϵ, δ)-regular measures on $Gr_k(L)$ for some k. Suppose $\delta \ll 1$. Write*

$$g_n = K(n)D(n)K'(n),$$

where $K(n)$ and $K'(n)$ are orthogonal relative to the standard basis $\{e_1, \dots e_m\}$, and $D(n) = \operatorname{diag}(d_1(n), \dots, d_m(n)\}$ with $d_1(n) \ge \dots d_m(n)$.

(a) *Suppose*

$$\frac{d_k(n)}{d_{k+1}(n)} \to \infty \tag{C.11}$$

Then, for any subsequential limit λ of $g_n\eta_n$ we have

$$K(n)\,\mathrm{span}\{e_1,\dots,e_k\} \to W, \tag{C.12}$$

and $\lambda(\{W\}) \geq 1-\delta$.

(b) *Suppose $g_n\eta_n \to \lambda$ where λ is some measure on Gr_k. Suppose also that there exists a subspace $W \in Gr_k(L)$ such that $\lambda(\{W\}) > 5\delta$. Then, as $n \to \infty$, (C.11) holds. As a consequence, by part (a), (C.12) holds and $\lambda(\{W\}) \geq 1-\delta$.*

Proof of (a). This statement is standard. Without loss of generality, $K'(n)$ is the identity (or else we replace η_n by $K'(n)\eta_n$). By our assumptions, for $j_1 < \cdots < j_k$,

$$\frac{\|g_n(e_{j_1}\wedge\cdots\wedge e_{j_k})\|}{\|g_n(e_1\wedge\cdots\wedge e_k)\|} \to 0 \quad \text{unless } j_i = i \text{ for } 1 \leq i \leq k.$$

Therefore, if $U \notin I(\mathrm{span}\{e_{k+1},\dots,e_m\})$, $g_nU \to K(n)\,\mathrm{span}\{e_1,\dots,e_k\}$. It follows from the (ϵ,δ)-regularity of η_n that any limit of $g_n\eta_n$ must give weight at least $1-\delta$ to W.

Proof of (b). This is similar to [GM, Lemma 3.9]. Suppose $d_k(n)/d_{k+1}(n)$ does not go to ∞. Then, there is a subsequence of the g_n (which we again denote by g_n) such that either $d_k(n)/d_{k+1}(n)$ After passing again to a subsequence, we may assume that $K(n) \to K_*$ and that for every j, either $d_j(n)/d_{j+1}(n)$ is bounded as $n \to \infty$ or $d_j(n)/d_{j+1}(n) \to \infty$ as $n \to \infty$. Also without loss of generality we may assume that $K'(n)$ is the identity (or else we replace η_n by $K'(n)\eta_n$).

Let $1 \leq s \leq k \leq r \leq m$ be such that s is as small as possible, r is as large as possible, and $d_j(n)/d_{j+1}(n)$ is bounded for $s \leq j \leq r-1$. Then, for $j_1 < \cdots < j_k$,

$$\frac{\|g_n(e_{j_1}\wedge\cdots\wedge e_{j_k})\|}{\|g_n(e_1\wedge\cdots\wedge e_k)\|} \to 0 \quad \text{unless } j_i = i \text{ for } 1 \leq i \leq s-1 \text{ and } s \leq j_i \leq r \text{ for } s \leq i \leq k. \tag{C.13}$$

Let

$$V_- = \mathrm{span}\{e_1,\dots,e_{s-1}\}, \qquad V_+ = \mathrm{span}\{e_1,\dots,e_r\}.$$

Then, in view of (C.13), for U such that $U \notin I(V_+^\perp)\cup I(V_-^\perp)$,

$$g_nU \to U' \qquad \text{where } K_*V_- \subset U' \subset K_*V_+.$$

Therefore, we must have $V_- \subset K_*^{-1}W \subset V_+$. Furthermore, for $U \notin I(V_+^\perp)\cup I(V_-^\perp)$,

$$\text{if } g_nU \to W \text{ then } U \in I(K_*^{-1}W \cap V_-^\perp + V_+^\perp).$$

But, since η_n is (ϵ,δ)-regular,

$$\eta_n(Nbhd_\epsilon(I(V_+^\perp)\cup I(V_-^\perp)\cup I(K_*^{-1}W\cap V_-^\perp + V_+^\perp))) < 3\delta.$$

Therefore $\lambda(\{W\}) < 3\delta$ which is a contradiction. Thus $d_k(n)/d_{k+1}(n) \to \infty$. Now by part (a) (C.12) holds, and $\lambda(\{W\}) \geq 1 - \delta$. $\square$

Let $\mathcal{F} = \mathcal{F}(L)$ denote the space of full flags on L. Let $\hat{X} = X \times \mathcal{F}$. The cocycle A satisfies the cocycle relation

$$A(g_1 g_2, x) = A(g_1, g_2 x) A(g_2, x).$$

The group $SL(2,\mathbb{R})$ acts on the space $\hat{X}$ is by

$$g \cdot (x, f) = (gx, A(g,x)f). \tag{C.14}$$

We choose some ergodic μ-stationary measure $\hat{\nu}$ on $\hat{X}$, which projects to ν, and write

$$d\hat{\nu}(x, f) = d\nu(x)\, d\eta_x(f).$$

Note that Lemma C.9 applies to the measures η_x on $\mathcal{F}$.

Lemma C.12 (Furstenberg)**.** *For $1 \leq s \leq \dim L$, let $\sigma_s : SL(2,\mathbb{R}) \times \hat{X} \to \mathbb{R}$ be given by*

$$\bar{\sigma}_s(g, x, f) = \log \frac{\|A(g,x)\xi_s(f)\|}{\|\xi_s(f)\|}$$

where $\xi_s(f)$ is the s-dimensional component of the flag f. Then, we have

$$\lambda_1 + \cdots + \lambda_s = \int_{SL(2,\mathbb{R})} \int_{\hat{X}} \bar{\sigma}_s(g, x, f)\, d\hat{\nu}(x, f)\, d\mu(g).$$

where λ_i is the i'th Lyapunov exponent of the cocycle A.

Proof. See the proof of [GM, Lemma 5.2]. $\square$

For $a \in \tilde{B}$, let the measures ν_a, $\hat{\nu}_a$ be as defined in [BQ, Lemma 3.2], i.e.

$$\nu_a = \lim_{n\to\infty} (a_n \dots a_1)_*^{-1} \nu$$

$$\hat{\nu}_a = \lim_{n\to\infty} (a_n \dots a_1)_*^{-1} \hat{\nu}.$$

The limits exist by the martingale convergence theorem. We disintegrate

$$d\hat{\nu}(x, f) = d\nu(x)\, d\eta_x(f), \qquad d\hat{\nu}_a(x, f) = d\nu_a(x)\, d\eta_{a,x}(f).$$

For $1 \leq k \leq m$, let $\eta_x^k = (\xi_k)_* \eta_x$ and $\eta_{a,x}^k = (\xi_k)_* \eta_{a,x}$. Then, η_x^k and $\eta_{a,x}^k$ are measures on $Gr_k(L)$.

If the Lyapunov spectrum is simple, we expect the measures $\eta_{a,x}$ to be supported at one point. In the general case, let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$$

denote the Lyapunov exponents, and let

$$I = \{1 \leq r \leq m-1 \ : \ \lambda_r = \lambda_{r+1}\}.$$

Then, by the multiplicative ergodic theorem, Lemma C.9 and Lemma C.11 (a), for $r \notin I$, we have $\eta_{a,x}^{m-r}$ is supported at one point. (This point is the part of the flag (C.1) corresponding to the Lyapunov exponents $\lambda_{r+1}, \dots, \lambda_m$.)

Claim C.13. *For any $r \in I$ and $\beta \times \nu$-almost all (a, x), for any subspace $W(a, x) \in Gr_{m-r}(L)$, we have $\eta_{a,x}^{m-r}(\{W(a, x)\}) = 0$.*

Proof of claim. Suppose there exists $\delta > 0$ so that for some $r \in I$ for a set (a, x) of positive measure, there exists $W(a, x) \subset Gr_{m-r}(L)$ with $\eta_{a,x}^{r}(\{W(a, x)\}) > \delta$. Then this happens for a subset of full measure by ergodicity.

We have for β-a.e. $a \in B$,

$$\hat{\nu}_a = \lim_{n\to\infty} (a_n \dots a_1)_*^{-1} \hat{\nu}.$$

Therefore, on a set of $\beta \times \nu$ full measure,

$$\lim_{n\to\infty} (a_n \dots a_1)_*^{-1} \eta_{a_n \dots a_1 x} = \eta_{a,x}.$$

Using (C.14), this means

$$\lim_{n\to\infty} A((a_n \dots a_1)^{-1}, a_n \dots a_1 x) \eta_{a_n \dots a_1 x} = \eta_{a,x}.$$

Note that by the cocycle relation,

$$A(g^{-1}, gx) = A(g, x)^{-1}.$$

Therefore,

$$A((a_n \dots a_1)^{-1}, a_n \dots a_1 x) = A(a_n \dots a_1, x)^{-1}.$$

Hence, on a set of $\beta \times \nu$-full measure,

$$\lim_{n\to\infty} A(a_n \dots a_1, x)^{-1} \eta_{a_n \dots a_1 x} = \eta_{a,x}.$$

In view of Lemma C.9 (cf. the proof of Lemma 14.4), there exists $\epsilon > 0$ and a compact $\mathcal{K}_\delta \subset X$ with $\nu(\mathcal{K}_\delta) > 1 - \delta$ such that the family of measures $\{\eta_x\}_{x \in \mathcal{K}_\delta}$ is uniformly $(\epsilon, \delta/5)$-regular. Let

$$\mathcal{N}_\delta(a, x) = \{n \in \mathbb{N} \ : \ a_n \dots a_1 x \in \mathcal{K}_\delta\}.$$

Write

$$A(a_n \dots a_1, x)^{-1} = K_n(a, x) D_n(a, x) K_n'(a, x) \tag{C.15}$$

where K_n and K_n' are orthogonal, and D_n is diagonal with non-increasing entries. We also write

$$A(a_n \dots a_1, x) = \bar{K}_n(a, x) \bar{D}_n(a, x) \bar{K}_n'(a, x). \tag{C.16}$$

where $\bar{K}_n$ and $\bar{K}'_n$ are orthogonal, and $\bar{D}_n$ is diagonal with non-increasing entries. Let $d_1(n,a,x) \geq \ldots \geq d_m(n,a,x)$ be the entries of $D_n(a,x)$, and let $\bar{d}_1(n,a,x) \geq \bar{d}_2(n,a,x) \geq \bar{d}_m(n,a,x)$ be the entries of $\bar{D}_n(a,x)$. Then,

$$\text{(C.17)} \quad \bar{d}_j(n,a,x) = d^{-1}_{m+1-j}(n,a,x), \qquad \bar{K}'_n(a,x) = w_0 K_n(a,x)^{-1} w_0^{-1}, \quad \bar{K}_n(a,x) = w_0 K'_n(a,x)^{-1} w_0,$$

where $w_0 = w_0^{-1}$ is the permutation matrix mapping e_j to e_{m+1-j}. Then, by Lemma C.11 (b), for $\beta \times \nu$ almost all (a,x), $\eta^{m-r}_{a,x}(\{W(a,x)\}) \geq 1-\delta$ (and thus $W(a,x)$ is unique) and as $n \to \infty$ along $\mathcal{N}_\delta(a,x)$ we have:

$$d_{m-r}(n,a,x)/d_{m+1-r}(n,a,x) \to \infty,$$

and

$$\text{(C.18)} \qquad K_n(a,x) \operatorname{span}\{e_1, \ldots, e_{m-r}\} \to W(a,x),$$

where the e_i are the standard basis for L. Then, by (C.17),

$$\text{(C.19)} \qquad \bar{d}_r(n,a,x)/\bar{d}_{r+1}(n,a,x) \to \infty,$$

and

$$\bar{K}'_n(a,x)^{-1} \operatorname{span}\{e_{r+1}, \ldots, e_m\} \to w_0 W(a,x)$$

Therefore for any $\epsilon_1 > 0$ there exists a subset $H_{\epsilon_1} \subset B \times X$ of $\beta \times \nu$-measure at least $1 - \epsilon_1$ such that the convergence in (C.19) and (C.18) is uniform over $(a,x) \in H_{\epsilon_1}$. Hence there exists $M > 0$ such that for any $(a,x) \in H_{\epsilon_1}$, and any $n \in \mathcal{N}_\delta(a,x)$ with $n > M$,

$$\text{(C.20)} \qquad \bar{K}'_n(a,x)^{-1} \operatorname{span}\{e_{r+1}, \ldots, e_m\} \in Nbhd_{\epsilon_1}(w_0 W(a,x)).$$

By Lemma C.9 (cf. the proof of Lemma 14.4) there exists a subset $H''_{\epsilon_1} \subset X$ with $\nu(H''_{\epsilon_1}) > 1 - c_2(\epsilon_1)$ with $c_2(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$ such that for all $x \in H''_{\epsilon_1}$, and any $U \in Gr_{m-r}(L)$,

$$\eta^r_x(Nbhd_{2\epsilon_1}(I(U))) < c_3(\epsilon_1),$$

where $c_3(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$. Let

$$\text{(C.21)} \quad H'_{\epsilon_1} = \{(a,x,f) \ : \ (a,x) \in H_{\epsilon_1}, \quad x \in H''_{\epsilon_1} \quad \text{and} \quad d(\xi_r(f), w_0 W(a,x)) > 2\epsilon_1\}.$$

Then, $(\beta \times \hat{\nu})(H'_{\epsilon_1}) > 1 - \epsilon_1 - c_2(\epsilon_1) - c_3(\epsilon_1)$, hence $(\beta \times \hat{\nu})(H'_{\epsilon_1}) \to 1$ as $\epsilon_1 \to 0$. Furthermore, by (C.20) and the definition of H'_{ϵ_1}, for $(a,x,f) \in H'_{\epsilon_1}$ and $n \in N_\delta(a,x)$ with $n > M$, we have

$$d(\xi_r(f), I(\bar{K}'_n(a,x)^{-1} \operatorname{span}\{e_{r+1}, \ldots, e_m\})) > \epsilon_1.$$

Therefore, in view of (C.16) there exists $C = C(\epsilon_1)$, such that for any $(a,x,f) \in H'_{\epsilon_1}$, any $n \in N_\delta(a,x)$ with $n > M$,

$$C > \|A(a_n \dots a_1, x)\xi_r(f)\| \prod_{i=1}^{r} \bar{d}_i(n, a, x)^{-1} > \frac{1}{C}, \tag{C.22}$$

(c.f [GM, Lemma 5.1]). Note that for all $(a, x, f) \in B \times \hat{X}$, all $n \in \mathbb{N}$ and and $j = r-1$ or $j = r+1$ we have

$$\|A(a_n \dots a_1, x)\xi_j(f)\| \leq \|A(a_n \dots a_1, x)\|_{\bigwedge^j(L)} \leq \prod_{i=1}^{j} \bar{d}_i(n, a, x). \tag{C.23}$$

Then, in view of (C.22) and (C.23), for all $(a, x, f) \in H'_{\epsilon_1}$, as $n \to \infty$ in $\mathcal{N}_\delta(a, x)$,

$$\log \frac{\|(A(a_n \dots a_1, x))\xi_r(f)\|^2}{\|(A(a_n \dots a_1, x))\xi_{r-1}(f)\|\|(A(a_n \dots a_1, x))\xi_{r+1}(f)\|} \geq \log \frac{\bar{d}_r(n, a, x)}{\bar{d}_{r+1}(n, a, x)} \to \infty \tag{C.24}$$

Since $(\beta \times \hat{\nu})(H'_{\epsilon_1}) \to 1$ as $\epsilon_1 \to 0$, (C.24) holds as $n \to \infty$ along $\mathcal{N}_\delta(a, x)$ for $\beta \times \hat{\nu}$ almost all $(a, x, f) \in B \times \hat{X}$.

For $1 \leq s \leq m$, let $\sigma_s : B \times \hat{X} \to \mathbb{R}$ be defined by $\sigma_s(a, x, f) = \bar{\sigma}_s(a_1, x, f)$, where $\bar{\sigma}$ is as in Lemma C.12. Then, the left hand side of (C.24) is exactly

$$\sum_{j=0}^{n-1} (2\sigma_r - \sigma_{r-1} - \sigma_{r+1})(\hat{T}^j(a, x, f)).$$

Also, we have $n \in \mathcal{N}_\delta(a, x)$ if and only if $\hat{T}^n(a, x) \in \mathcal{K}_\delta$. Then, by Lemma C.7,

$$\int_{B \times \hat{X}} (2\sigma_r - \sigma_{r-1} - \sigma_{r+1})(q)\, d(\beta \times \hat{\nu})(q) > 0.$$

By Furstenberg's formula Lemma C.12, the left hand side of the above equation is $\lambda_r - \lambda_{r+1}$. Thus $\lambda_r > \lambda_{r+1}$, contradicting our assumption that $r \in I$. This completes the proof of the claim. □

Proof of Theorem C.4. Consider the (partial) Lyapunov flag (C.1), and let $s_i = \dim \hat{V}_i$. Then, $\eta^{s_i}_{a,x}$, which is the projection of $\eta_{a,x}$ from $\mathcal{F}(L)$ to $Gr_{s_i}(L)$ is the delta measure at $\hat{V}_i(a, x)$.

Let $\mathcal{F}(s_1, \dots, s_{k-1})$ denote the space of partial flags with subspaces of dimensions s_i. Let Y denote the fiber of the projection $\mathcal{F} \to \mathcal{F}(s_1, \dots, s_{k-1})$. Then, Y is isomorphic to $\bigoplus_{i=1}^{k} \mathcal{F}(\hat{V}_i/\hat{V}_{i-1})$. Let $\bar{\eta}^{(i)}_{a,x}$ be the restriction of $\eta_{a,x}$ to $\mathcal{F}(\hat{V}_i/\hat{V}_{i-1})$. Then, by Claim C.13, $\bar{\eta}^{(i)}$ has no atoms on $\mathcal{F}(\hat{V}_i/\hat{V}_{i-1})$, and also, for $1 \leq j < \dim(\hat{V}_i/\hat{V}_{i-1})$, the projection of $\bar{\eta}^{(i)}$ to any $Gr_j(\hat{V}_i/\hat{V}_{i-1})$ has no atoms. It follows that for every $\delta > 0$ there exists $\epsilon = \epsilon(\delta) > 0$ and a subset $\mathcal{K}_1 = \mathcal{K}_1(\delta) \subset B \times X$ with $(\beta \times \nu)(\mathcal{K}_1) > 1 - \delta$ such that for all $(a, x) \in \mathcal{K}_1$, $\bar{\eta}^{(i)}_{a,x}$ is (ϵ, δ)-regular.

Let A_i be the restriction of the cocycle A to $\hat{V}_i/\hat{V}_{i-1}$. Note that

$$\bar{\eta}^{(i)}_{\hat{T}^n(a,x)} = A_i(a_n \dots a_1, x)\bar{\eta}^{(i)}_{a,x}$$

Then by Lemma C.11 (i) there exists $C = C(\delta)$ such that for all $(a,x) \in \mathcal{K}_1$ and all n with $\hat{T}^n(a,x) \in \mathcal{K}_1$ we have $\|A_i(n,a,x)\| \le C(\delta)$. It follows that for all $n \in \mathbb{Z}$

$$(\beta \times \nu)(\{(a,x) \in B \times X \;\;:\;\; \|A_i(n,a,x)\| > C(\delta)\}) \le 2\delta.$$

Since $\delta > 0$ is arbitrary, this means (by definition) that the cocycle A_i is bounded in the sense of Schmidt, see [Sch]. It is proved in [Sch] that any bounded cocycle is conjugate to a cocycle taking values in an orthogonal group. □

Proof of Theorem C.5. As in §14, let $\tilde{B}$ be the space of by-infinite sequences of elements of $SL(2,\mathbb{R})$, and we consider the two-sided random walk as a shift map on $\tilde{B} \times X$. We denote a point in $\tilde{B}$ by $a \vee b$ where a denotes the "future" of the trajectory and b denotes the "past". Then, at almost all points $(a \vee b, x)$ we have both the flags (C.1) and (C.3). The two flags are generically in general position (see e.g. [GM, Lemma 1.5]) and thus we can intersect the flags to define the (shift-invariant) Lyapunov subspaces $\mathcal{V}_i(a \vee b, x)$ so that

$$V_i(b,x) = \bigoplus_{j=1}^{i} \mathcal{V}_j(a \vee b, x), \qquad \hat{V}_i(a,x) = \bigoplus_{j=m+1-i}^{m} \mathcal{V}_j(a \vee b, x).$$

Then

$$V_i(b,x)/V_{i-1}(b,x) \cong \mathcal{V}_i(a \vee b, x) \cong \hat{V}_{m+1-i}(a,x)/\hat{V}_{m-i}(a,x).$$

Since we already showed that the restriction of the cocycle to each $\hat{V}_{m+1-i}/\hat{V}_{m-i}$ is conjugate into an orthogonal group, it follows that the same is true of the restriction of the cocycle to $\mathcal{V}_i$ or to V_i/V_{i-1}. □

References

[ABEM] J. Athreya, A. Bufetov, A. Eskin and M. Mirzakhani. Lattice Point Asymptotics and Volume Growth on Teichmüller space. arXiv:math/0610715v3 [math.DS]

[At] G. Atkinson. Recurrence of co-cycles and random walks. *J. London Math. Soc.* (2) **13** (1976), no. 3, 486–488.

[Ath] J. Athreya. Quantitative recurrence and large deviations for Teichmüller geodesic flow. *Geom. Dedicata* **119** (2006), 121–140.

[AthF] J. Athreya, G. Forni. Deviation of ergodic averages for rational polygonal billiards. *Duke Math. J.* **144** (2008), no. 2, 285-319.

[ACO] L. Arnold, N. Cong, V. Osceledets. "Jordan Normal Form for Linear Cocycles". *Random Operators and Stochastic Equations* **7** (1999), 301-356.

[AEZ] J. Athreya, A. Eskin, A. Zorich, Rectangular billiards and volumes of spaces of quadratic differentials, *In preparation.*

[AEM] A. Avila, A. Eskin, M. Moeller. Symplectic and Isometric SL(2,R) invariant subbundles of the Hodge bundle. `arXiv:1209.2854 [math.DS]` (2012).

[AF] A. Avila, G. Forni. Weak mixing for interval exchange transformations and translation flows. *Ann. of Math.* (2) **165** (2007), no. 2, 637 664.

[AG] A. Avila, S. Gouëzel. Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmller flow. `arXiv:1011.5472 [math.DS]` (2010).

[AGY] A. Avila, S. Gouëzel, J-C. Yoccoz. Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.* No. 104 (2006), 143–211.

[AV] A. Avila, M. Viana. Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture, *Acta Math.* **198** , No. 1, (2007), 1–56.

[Ba] M. Bainbridge. Billiards in L-shaped tables with barriers, `arXiv:0705.3271` (2008), 1–63.

[BaM] M. Bainbridge and M. Möller. Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus three. `arXiv:0911.4677v1 [math.AG]` (2009)

[BoM] I. Bouw and M. Möller. Teichmller curves, triangle groups, and Lyapunov exponents. *Ann. of Math.*, to appear.

[BG] A. Bufetov, B. Gurevich. Existence and Uniqueness of the Measure of Maximal Entropy for the Teichmüller Flow on the Moduli Space of Abelian Differentials. arXiv:math/0703020v3 [math.DS].

[BQ] Y. Benoist and J-F Quint. Mesures Stationnaires et Fermés Invariants des espaces homogènes. *Preprint.*

[Ca] K. Calta. Veech surfaces and complete periodicity in genus two. *J. Amer. Math. Soc.* **17** (2004), no. 4, 871–908

[CK] V. Climenhaga, A. Katok. Measure theory through dynamical eyes. arXiv:1208.4550 [math.DS]

[CM] D. Chen, M. Möeller. Non-varying sums of Lyapunov exponents of Abelian differentials in low genus. *Preprint.*

[CW] K. Calta, K. Wortman. On unipotent flows in H(1,1). Ergodic Theory Dynam. Systems 30 (2010), no. 2, 379–398.

[Dan1] S.G. Dani, On invariant measures, minimal sets and a lemma of Margulis, *Invent. Math.* **51** (1979), 239–260.

[Dan2] S.G. Dani, Invariant measures and minimal sets of horoshperical flows, *Invent. Math.* **64** (1981), 357–385.

[Dan3] S.G. Dani, On orbits of unipotent flows on homogeneous spaces, *Ergod. Theor. Dynam. Syst.* **4** (1984), 25–34.

[Dan4] S.G. Dani, On orbits of unipotent flows on homogenous spaces II, *Ergod. Theor. Dynam. Syst.* **6** (1986), 167–182.

[DHL] V. Delecroix, P. Hubert, S. Lelievre. Diffusion for the periodic wind-tree model. *Preprint.*

[DM1] S.G. Dani and G.A. Margulis, Values of quadratic forms at primitive integral points, *Invent. Math.* **98** (1989), 405–424.

[DM2] S.G. Dani and G.A. Margulis, Orbit closures of generic unipotent flows on homogeneous spaces of $SL(3,\mathbb{R})$, *Math. Ann.* **286** (1990), 101–128.

[DM3] S.G. Dani and G.A. Margulis. Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces, *Indian. Acad. Sci. J.* **101** (1991), 1–17.

[DM4] S.G. Dani and G.A. Margulis, Limit distributions of orbits of unipotent flows and values of quadratic forms, in: *I. M. Gelfand Seminar*, Amer. Math. Soc., Providence, RI, 1993, pp. 91–137.

[Ef] Effros, Edward G. Transformation groups and C^*-algebras. *Ann. of Math.* (2) **81** (1965) 3855.

[EKL] M. Einsiedler, A. Katok, E. Lindenstrauss. Invariant measures and the set of exceptions to Littlewood's conjecture. *Ann. of Math.* (2) **164** (2006), no. 2, 513–560.

[EKZ] A. Eskin, M. Kontsevich, A. Zorich. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow. *Preprint.*

[EL] M. Einsiedler, E. Lindenstrauss. “Diagonal actions on locally homogeneous spaces.” In Homogeneous flows, moduli spaces and arithmetic, 155241, Clay Math. Proc., 10, Amer. Math. Soc., Providence, RI, 2010.

[EMa] A. Eskin, H. Masur. Asymptotic formulas on flat surfaces. *Ergodic Theory and Dynamical Systems,* **21** (2) (2001), 443–478.

[EMM] A. Eskin, J. Marklof, D. Morris. Unipotent flows on the space of branched covers of Veech surfaces. *Ergodic Theory Dynam. Systems* **26** (2006), no. 1, 129–162.

[EMM1] A. Eskin, G. Margulis, and S. Mozes. “Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture.” *Ann. of Math. (2)* **147** (1998), no. 1, 93–141.

[EMM2] A. Eskin, G. Margulis, and S. Mozes. “Quadratic forms of signature $(2, 2)$ and eigenvalue spacings on flat 2-tori.” *Ann. of Math. (2)* **161** (2005), no. 2, 679–725.

[EMiMo] A. Eskin, M. Mirzakhani and A. Mohammadi. Isolation Theorems for $SL(2,\mathbb{R})$-invariant submanifolds in Moduli space. *In preparation.*

[EMR] A. Eskin, M. Mirzakhani and K. Rafi. Counting closed geodesics in strata. *Preprint.*

[EMS] A. Eskin, H. Masur, M. Schmoll, Billiards in rectangles with barriers. *Duke Math. J.* **118** No. 3 (2003), 427–463.

[EMZ] A. Eskin, H. Masur, A. Zorich, Moduli spaces of Abelian differentials: the principal boundary, counting problems and the Siegel–Veech constants. *Publications Mathématiques de l'IHÉS*, **97** (1) (2003), 61–179.

[EMat] A. Eskin, C. Matheus. Semisimplicity of the Laypunov spectrum for irreducible cocycles. *In preparation.*

[EO] A. Eskin, A. Okounkov. *Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials.* Invent. Math. **145** (2001), no. 1, 59–103.

[EhEh] P. Ehrenfest and T. Ehrenfest. Begriffliche Grundlagen der statistischen Auffassung in der Mechanik Encykl. d. Math. Wissensch. IV 2 II, Heft 6, 90 S (1912) (in German, translated in:) The conceptual foundations of the statistical approach in mechanics, (trans. Moravicsik, M. J.), 10-13 Cornell Univer- sity Press, Itacha NY, (1959).

[Fo] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus.* Annals of Math., **155**, no. 1, 1–103 (2002)

[Fo2] G. Forni. On the Lyapunov exponents of the Kontsevich-Zorich cocycle. *Handbook of dynamical systems. Vol. 1B*, pp 549–580. Elsevier B. V., Amsterdam, 2006.

[FoM] G. Forni, C. Matheus, *An example of a Teichmüller disk in genus 4 with degenerate Kontsevich-Zorich spectrum*, `arXiv:0810.0023` (2008), 1–8.

[FoMZ] G. Forni, C. Matheus, A. Zorich. ”Lyapunov Spectrum of Invariant Subbundles of the Hodge Bundle”. *Preprint.*

[Fu] A. Furman. “Random walks on groups and random transformations.” *Handbook of dynamical systems*, Vol. 1A, 931 - 1014, North-Holland, Amsterdam, 2002.

[F1] H. Furstenberg. A poisson formula for semi-simple Lie groups. *Ann. of Math.* **77**(2), 335 - 386 (1963).

[F2] H. Furstenberg. Non commuting random products. *Trans. Amer. Math. Soc.* **108**, 377 -428 (1963).

[GM] I.Ya. Gol'dsheid and G.A. Margulis. Lyapunov indices of a product of random matrices. *Russian Math. Surveys* **44:5** (1989), 11-71.

[GR1] Y. Guivarc'h and A. Raugi, Frontiere de Furstenberg, propriotes de contraction et theoremes de convergence, *Z. Wahrsch. Verw. Gebiete* **69** (1985), 187-242.

[GR2] Y. Guivarc'h and A. Raugi. Propriétés de contraction d'un semi-groupe de matrices inversibles. Coefficients de Liapunoff d'un produit de matrices aléatoires indépendantes. (French) [Contraction properties of an invertible matrix semigroup. Lyapunov coefficients of a product of independent random matrices] *Israel J. Math.* **65** (1989), no. 2, 165196.

[HLM] P. Hubert, E. Lanneau, M. Möller. $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures via topological splittings. Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces, 145 169, Surv. Differ. Geom., 14, Int. Press, Somerville, MA, 2009.

[HLT] P. Hubert, S. Lelievre, S. Troubetzkoy. The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion. *Preprint.*

[HST] P. Hubert, M. Schmoll and S. Troubetzkoy, Modular fibers and illumination problems. *Int. Math. Res. Not. IMRN* **2008**, no. 8,

[Ke] H. Kesten. Sums of stationary sequences cannot grow slower than linearly. *Proc. Amer. Math. Soc.* **49** (1975), 205–211.

[Kn] A. Knapp. Lie Groups, Beyond an Introduction. Second Edition. Progress in Mathematics Vol. **140**, Birkhäuser, Boston, 2002.

[KH] A. Katok, B. Hasselblat Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press, 1995.

[LS] F. Ledrapier and J. M. Strelcyn. A proof of the estimation from below in Pesin's entropy formula. *Ergodic Theory Dyn. Syst.* **2** (1982), 203–219.

[LY] F. Ledrapier, and L. S. Young: The metric entropy of diffeomorphisms. I. *Ann. Math.* **122** (1985), 503–539.

[M1] R. Mañé. A proof of Pesin's formula. *Ergodic Theory Dyn. Syst.* **1** (1981), 95–102.

[M2] R. Mañé. Ergodic Theory and Differentiable Dynamics. Springer-Verlag 1987.

[Mar1] G.A. Margulis, *On the action of unipotent groups in the space of lattices*, In *Lie Groups and their representations*, Proc. of Summer School in Group Representations, Bolyai Janos Math. Soc., Akademai Kiado, Budapest, 1971, p. 365–370, Halsted, New York, 1975.

[Mar2] G.A. Margulis, Formes quadratiques ind'efinies et flots unipotents sur les spaces homog'enes, *C. R. Acad. Sci. Paris Ser. I* **304** (1987), 247–253.

[Mar3] G.A. Margulis, em Discrete Subgroups and Ergodic Theory, in: *Number theory, trace formulas and discrete subgroups*, a symposium in honor of A Selberg, pp. 377–398. Academic Press, Boston, MA, 1989.

[Mar4] G.A.Margulis, *Indefinite quadratic forms and unipotent flows on homogeneous spaces*, In *Dynamical systems and ergodic theory*, Vol. 23, pp. 399–409, Banach Center Publ., PWN – Polish Scientific Publ., Warsaw, 1989.

[MaT] G. A. Margulis and G. M. Tomanov. Invariant measures for actions of unipotent groups over local fields on homogeneous spaces. *Invent. Math.* **116** (1994), 347–392.

[Mas1] H. Masur. Interval exchange transformations and measured foliations. *Ann. of Math.* (2) **115** (1982), no. 1, 169–200.

[Mas2] H. Masur. The growth rate of trajectories of a quadratic differential. *Ergodic Theory Dynam. Systems* **10** (1990), 151–176.

[Mas3] H. Masur. Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential. In *Holomorphic Functions and Moduli*, Vol. 1, D. Drasin, ed., Springer-Verlag: New York, 1988, pp. 215–228.

[Mc1] C. McMullen. Billiards and Teichmller curves on Hilbert modular surfaces. *J. Am. Math. Soc.* **16**, 857–885 (2003)

[Mc2] C. McMullen. Teichmller geodesics of infinite complexity. *Acta Math.* **191**, 191–223 (2003)

[Mc3] C. McMullen. Teichmller curves in genus two: Discriminant and spin. *Math. Ann.* **333**, 87–130 (2005)

[Mc4] C. McMullen. Teichmller curves in genus two: The decagon and beyond. *J. Reine Angew. Math.* **582**, 173–200.

[Mc5] C. McMullen. Teichmller curves in genus two: torsion divisors and ratios of sines. *Invent. Math.* **165** (2006), no. 3, 651–672.

[Mc6] C. McMullen. Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two. *Ann. of Math.* (2) **165** (2007), no. 2, 397–456.

[Mö1] M. Möller. Variations of Hodge structures of a Teichmller curve. *J. Amer. Math. Soc.* **19** (2006), no. 2, 327–344.

[Mö2] M. Möller. Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmller curve. *Invent. Math.* **165** (2006), no. 3, 633–649.

[Mö3] M. Möller. Finiteness results for Teichmller curves. *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 1, 63–83.

[Mö4] M. Möller, Linear manifolds in the moduli space of one-forms, *Duke Math. J.* **144** No. 3 (2008), 447–488.

[Mor] D. W. Morris. *Ratner's Theorems on Unipotent Flows.* Univ. of Chicago Press: Chicago, 2005. http://arxiv.org/math.DS/0310402

[Moz] S. Mozes. Epimorphic subgroups and invariant measures. *Ergodic Theory Dynam. Systems* **15** (1995), no. 6, 1207–1210.

[MoSh] S. Mozes, N. Shah. On the space of ergodic invariant measures of unipotent flows. *Ergodic Theory Dynam. Systems* **15** (1995), no. 1, 149 -159.

[MZ] R. Zimmer, D. Witte Morris. Ergodic theory, groups, and geometry. CBMS Regional Conference Series in Mathematics, 109. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2008. x+87 pp. ISBN: 978-0-8218-0980-8

[NZ] A. Nevo and R. Zimmer. Homogeneous Projective Factors for actions of semisimple Lie groups. *Invent. Math.* **138** (1999), no. 2, 229–252.

[Ra1] M. Ratner. Rigidity of horocycle flows. *Ann. of Math.* **115** (1982), 597–614.

[Ra2] M. Ratner. Factors of horocycle flows. *Ergodic Theory Dynam. Systems* **2** (1982), 465–489.

[Ra3] M. Ratner. Horocycle flows, joinings and rigidity of products. *Ann. of Math.* **118** (1983), 277–313.

[Ra4] M. Ratner. Strict measure rigidity for unipotent subgroups of solvable groups. *Invent. Math.* **101** (1990), 449–482.

[Ra5] M. Ratner. On measure rigidity of unipotent subgroups of semisimple groups. *Acta Math.* **165** (1990), 229–309.

[Ra6] M. Ratner. On Raghunathan's measure conjecture. *Ann. of Math.* **134** (1991), 545–607.

[Ra7] M. Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.* **63** (1991), no. 1, 235–280.

[R] V.A. Rokhlin. Lectures on the theory of entropy of transformations with invariant measures. *Russ. Math. Surv.* **22:5** (1967), 1–54.

[Sch] K. Schmidt. Amenability, Kazhdan's property T, strong ergodicity and invariant means for ergodic group-actions. *Ergodic Theory Dynamical Systems* **1** (1981), no. 2, 223–236.

[S1] R. E. Schwartz. Obtuse triangular billiards. I. Near the $(2, 3, 6)$(2,3,6) triangle. *Experiment. Math.* **15** (2006), no. 2, 161 182.

[S2] R. E. Schwartz. Obtuse triangular billiards. II. One hundred degrees worth of periodic trajectories. *Experiment. Math.* **18** (2009), no. 2, 137 171.

[Ve1] W. Veech, Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math.*, **15** (1982), 201–242.

[Ve2] W. Veech. Siegel measures. *Ann. of Math.* **148** (1998), 895–944.

[Wr] A. Wright. The field of definition of affine invariant submanifolds of the moduli space of abelian differentials. arXiv:1210.4806[math.DS]

[ZK] A.N. Zemlyakov, A.B. Katok. *Topological transitivity of billiards in polygons. Matem. Zametki* **18**(2) (1975) pp. 291–300. English translation in *Math. Notes* **18**(2) (1976) pp. 760–764.

[Zi1] R. J. Zimmer, Induced and amenable ergodic actions of Lie groups, *Ann. Sci. Ecole Norm. Sup.* **11** (1978), no. 3, 407 428.

[Zi2] R. J. Zimmer. *Ergodic Theory and Semisimple Groups.* Birkhäuser: Boston, 1984.

[Zo] A. Zorich, A. Flat Surfaces. *Frontiers in Number Theory, Physics, and Geometry. I. Berlin: Springer, 2006.* 437–583.

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